Crash course in applied QFT

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Outline:

0) Conventions

1) Relativistic wave equations

2) Interacting quantum field theories

0) Conventions

0.1) Units

In SI: Speed of light in vacuum c = 299 792 458 m/s (exact)

In natural units: c = 1 (0.1) Establishes relation between units of length and units of time.

Similarly: In SI, Planck's constant $\hbar \equiv \frac{h}{2\pi} = 1.054\ldots \cdot 10^{-34}\,\mathrm{Js}$

$$\equiv 1 \quad (0.2)$$

Establishes a relation between units of time and units of energy.

Also: $E = mc^2$, $c = 1$ $\Rightarrow$ [energy] = [mass]

$\Rightarrow$ can express length, time as inverse energies! (units)

$m_e = 0.511\,\mathrm{MeV}$, $m_\mu = 106\,\mathrm{MeV}$, $m_p = 940\,\mathrm{MeV}$, ...

Momentum $|\vec{p}| = \frac{|\vec{v}|\,m}{\sqrt{1 - v^2/c^2}}$: [momentum] = [energy]

For ultra-relativistic particles (photons): $|\vec{p}| = E$

$\hbar c = 197\,\mathrm{MeV} \cdot \mathrm{fm} \simeq 1$

$\hookrightarrow 10^{-15}\,\mathrm{m} = 10^{-13}\,\mathrm{cm} = 1$ Fermi

$$\Rightarrow \frac{1}{\mathrm{fm}} = 197\,\mathrm{MeV} \simeq \frac{1}{5}\,\mathrm{GeV} \quad (0.3)$$

$$\Rightarrow \frac{1}{m} = 0.2\cdot 10^{-6}\ eV\ ,\quad 1m \simeq \frac{5\cdot 10^{15}}{GeV}$$

$$1s = \frac{1.5\cdot 10^{24}}{GeV} \qquad (0.4)$$

Decay width $\Gamma = 1/\text{lifetime}$; $[\Gamma] = [E]$

$$\left.\begin{aligned} \tau = 1s \Leftrightarrow \Gamma &= \tfrac{2}{3}\cdot 10^{-24}\ GeV \\ \Gamma = 1\ GeV \Leftrightarrow \tau &\simeq \tfrac{2}{3}\cdot 10^{-24}\ s \end{aligned}\right\} \quad (0.5)$$

Cross section: has units of area

$$1\ b\ (\text{barn}) \equiv 10^{-24}\ cm^2 = 100\ fm^2 = \frac{1}{3.88\cdot 10^{-4}\ GeV^2}$$

$$\Rightarrow 1\ pb = 10^{-12}\ b = 10^{-36}\ cm^2 = \frac{1}{3.88\cdot 10^{8}\ GeV^2}$$

$$\Rightarrow \frac{1}{GeV^2} = 3.88\cdot 10^{8}\ pb \qquad (0.6)$$

0.2) Lorentz invariance

Are often interested in (ultra-)relativistic particles: use Lorentz covariant description!

4-vector: $A^\mu = (A_0, \vec{A}) \qquad (0.7)$

e.g. 4-momentum $p^\mu = (E, \vec{p}) \qquad (0.8)$

Spacetime coordinate : $x^\mu = (t, \vec{x}) \qquad (0.9)$

$$A^\mu = g^{\mu\nu} A_\nu \equiv \sum_{\nu=0}^{3} g^{\mu\nu} A_\nu \quad \text{(summation convention)}$$

with Minkowski metric $g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ (0.10)

Spacetime derivative: $\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$ (0.11)

4-momentum operator: $\hat{p}^\mu = +i\partial^\mu$ (0.12)

Lorentz trafo: $A'_\mu = \Lambda^\nu{}_\mu A_\nu$, $\det \Lambda = 1$ (0.13)

E.g. boost in x-direction w/ velocity $v = \beta c$:

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad (0.14)$$

Scalar product $A \cdot B \equiv A_\mu B^\mu = A^\mu B_\mu = g^{\mu\nu} A_\mu B_\nu$
$= g_{\mu\nu} A^\mu B^\nu$ (0.15)

is Lorentz invariant

E.g. two 4-momenta: $p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2$
$= E_1 E_2 - |\vec{p}_1||\vec{p}_2| \cos\theta_{12}$ (0.16)

$p \cdot x = Et - \vec{p} \cdot \vec{x}$: Argument of plane wave (in natl. units)

1) Relativistic wave equations (w/o interactions)

Need to respect relation between energy and 3-momentum:

$$E^2 - \vec{p}^{\,2} = m^2 \qquad (1.1)$$

Since $E \to i\,\partial/\partial t$, $\vec{p} \to -i\vec{\nabla}$: Every relativistic wave fct. has to satisfy $\partial^\mu \partial_\mu \Psi \equiv \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\Psi = -m^2 \Psi$ (1.2): Klein-Gordon eq.

If this is the only condition: Ψ is scalar (spin-0) field, $\Psi \equiv \phi$

Solution: $\phi = N e^{-ip\cdot x} = N e^{-i(Et - \vec{p}\cdot\vec{x})}$ (1.3) $\phi \in \mathbb{C}$

$$\Rightarrow \left.\begin{array}{l} \frac{\partial^2 \phi}{\partial t^2} = -E^2 \phi \\ \vec{\nabla}^2 \phi = -\vec{p}^{\,2}\phi \end{array}\right\} \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\phi = \left(-E^2 + \vec{p}^{\,2}\right)\phi \underset{(1.2)}{\overset{!}{=}} -m^2 \phi$$

$$\Rightarrow E^2 - \vec{p}^{\,2} = m^2 \quad \checkmark\checkmark$$

But: allows $E = \pm\sqrt{\vec{p}^{\,2} + m^2}$: $E < 0$ possible?! (1.4)

Probability density, as for Schrödinger eq:

$$\left.\begin{array}{l} \phi^* \frac{\partial^2 \phi}{\partial t^2} - \phi^* \vec{\nabla}^2 \phi + \phi^* m^2 \phi = 0 \\ \phi \frac{\partial^2 \phi^*}{\partial t^2} - \phi \vec{\nabla}^2 \phi^* + \phi m^2 \phi^* = 0 \end{array}\right\} -$$

$$\phi^* \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^*}{\partial t^2} - \phi^* \vec{\nabla}^2 \phi + \phi \vec{\nabla}^2 \phi^* = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left[\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right] - \vec\nabla\left(\phi^*\vec\nabla\phi - \phi\vec\nabla\phi^*\right) = 0$$

$$\Rightarrow \partial^\mu j_\mu = 0;\quad j_\mu = \Big(\underbrace{i\left[\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right]}_{\rho},\ \underbrace{i\left[\phi^*\vec\nabla\phi - \phi\vec\nabla\phi^*\right]}_{\vec j}\Big) \qquad (1.5)$$

Continuity eq. for conserved current

For $\phi = N\, e^{-i(Et - \vec p\cdot\vec x)}$ (1.6) $\Rightarrow \rho = 2|N|^2 E$ (1.7)

$\rho < 0$ if $E < 0$; Cannot be probability density!

(Can be charge density!) $\Rightarrow$ K-G eq. abandoned!

Dirac: Construct eq. leading to positive prob. density, by "taking the square root" of K-G eq., i.e. via 1st order differential eq.:

$$i\partial_\mu\gamma^\mu\psi - m\psi = 0 \qquad (1.8)$$

γ^μ: 4 constant (!) coefficients, but not $\mathbb{C}$-numbers: do not commute

Dirac algebra

ψ has to satisfy K-G eq., so that $E^2 = \vec p^{\,2} + m^2$

Operate with $\gamma^0\frac{\partial}{\partial t}$ on (1.8), use (1.8) for terms linear in $\partial/\partial t$, compare w/ K-G eq. $\Rightarrow$ need $\boxed{\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\cdot 1\!\!1}$ (1.9)

Smallest rep. for γ^μ in $D=4$ space-time dimensions uses 4×4 matrices $\Rightarrow \psi$ is 4-component spinor.

Often needs $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$