Representations of the Dirac algebra

1) Dirac representation : Useful for non-relativistic limit

$$\gamma^0 = \begin{pmatrix} 1\!\!1_{2\times 2} & 0 \\ 0 & -1\!\!1_{2\times 2} \end{pmatrix} \; ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad (1.10)$$

Pauli matrices

$$\Rightarrow \gamma_5 = \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}$$

Solutions of Dirac eq.: In terms of 2-spinor

$$\psi = \omega \cdot e^{-ip\cdot x} \; ; \quad \omega = \sqrt{|E|+m} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \qquad (1.11)$$

$$(1.10) \Rightarrow E\phi - \vec{p}\cdot\vec{\sigma}\,\chi = m\phi$$

$$-E\chi + \vec{p}\cdot\vec{\sigma}\,\phi = m\chi \Rightarrow \chi = \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi \qquad (1.12)$$

KG-eq. satisfied $\Rightarrow E^2 = \vec{p}^{\,2} + m^2$ automatic, but $E<0$ allowed

$\underline{E>0}$: In rest frame of particle : $\vec{p}=0$, $E=+m \Rightarrow \underline{\chi=0}$

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \text{describes 2 spin states} \qquad (1.13)$$

For $\vec{p} \neq 0$:
$$\omega = N \begin{pmatrix} \phi \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi \end{pmatrix} \equiv u(p) \qquad (1.14)$$

"small" component

Uses: $\vec{p}\cdot\vec{\sigma} \; \vec{p}\cdot\vec{\sigma} = \vec{p}^{\,2} \cdot 1\!\!1_{2\times 2}$

Dirac eq. in mom. space for $E>0$ solution: $(\not{p} - m)\, u(p) = 0$ (1.15)

$\not{p} \equiv p_\mu \gamma^\mu$

$\underline{E<0}$: In rest frame, $\vec{p}=0$, $E=-m \Rightarrow \phi = 0$

$\chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \chi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (1.16) 2 spin state

$\vec{p}\,' = -\vec{p}$

$p_0' = -p_0 = +E$

For $\vec{p} \neq 0$: $\omega = N \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E-m}\chi \\ \chi \end{pmatrix} = N \begin{pmatrix} \frac{-\vec{p}\cdot\vec{\sigma}}{|E|+m}\chi \\ \chi \end{pmatrix} \equiv v(p')$

$$v(p') = N \begin{pmatrix} \frac{\vec{p}\,'\cdot\vec{\sigma}}{p_0'+m}\chi \\ \chi \end{pmatrix} \quad (1.17)$$

$\psi = e^{ip'\cdot x}\, v(p')$; $p' = (-p^0, -\vec{p})$: $p'^0 = -E > 0$

Satisfies $(\not{p}'+m)\, v(p') = 0$ (1.18)

Note: γ_5 mixes small & large components

2) Chiral rep. of Dirac algebra: Good for ultra-relat. particles

$$\gamma^0 = \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix} ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \Rightarrow \gamma_5 = \begin{pmatrix} -1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix} \quad (1.18)$$

$$\text{or: } \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} ; \quad \sigma^\mu = (1\!\!1, \vec{\sigma}); \quad \bar{\sigma}^\mu = (1\!\!1, -\vec{\sigma}) \quad (1.19)$$

$$\Rightarrow p\cdot\sigma \; p\cdot\bar{\sigma} = p^2 = m^2 \quad (1.20)$$

Solutions of the Dirac eq.:

$$\psi = \begin{cases} u(p)\, e^{-ip\cdot x} & , E > 0 \\ v(p')\, e^{ip'\cdot x} & , E < 0 \;\; (p_0' > 0) \end{cases} \quad (1.20)$$

$u = \begin{pmatrix} u_- \\ u_+ \end{pmatrix}, \quad v = \begin{pmatrix} v_- \\ v_+ \end{pmatrix}$ ← left-handed / ← right-handed helicity

$$u(p,\lambda)_\pm = \omega_{\pm\lambda}(p)\, \chi_\lambda(p) \; ; \; v(p,\lambda)_\pm = \pm\, \omega_{\mp\lambda}(p)\, \chi_{-\lambda}(p) \quad (1.21)$$

↳ $\lambda = \pm 1$: labels spin

χ_λ are eigenstate of helicity: $\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi_\lambda(p) = \lambda\chi_\lambda(p)$ (1.23)

$\omega_\pm(p) = \sqrt{p_0 \pm |\vec{p}|}$, $p_0 > 0$; $\omega_- \ll \omega_+$ for $E \gg m$

For $p_z \neq -|\vec{p}|$:

$$\chi_+(p) = \underbrace{\frac{1}{[2|\vec{p}|(|\vec{p}|+p_z)]^{\frac{1}{2}}}}_{N(p)} \cdot \begin{pmatrix} |\vec{p}|+p_z \\ p_x + ip_y \end{pmatrix}; \quad \chi_-(p) = N(p)\cdot\begin{pmatrix} -p_x + ip_y \\ |\vec{p}| + p_z \end{pmatrix}$$

For $|\vec{p}| = -p_z$: $\chi_+(p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\chi_-(p) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ (1.24)

For $|E| \gg m$: only $u(p,\lambda)_\lambda$ and $v(p,\lambda)_{-\lambda}$ survive!

For $|\vec{p}| \to 0$: $\omega_\pm \to \sqrt{m}$

Probability current: $j_\mu \equiv \bar{\psi}\gamma_\mu\psi$, with $\bar{\psi} = \psi^\dagger\gamma^0$ (1.27)

herm. conj.

(For matrix M: $(M^\dagger)_{ij} = M^*_{ji}$)

Satisfies $\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$ (→ HW)

$$\rho = \bar{\psi}\gamma^0\psi \underset{(1.27)}{=} \psi^\dagger\underbrace{\gamma^0\gamma^0}_{1\!\!1}\psi = \underline{\underline{\psi^\dagger\psi > 0}}!$$

(1.29)

$$j^i = \bar{\psi}\gamma^i\psi = \psi^\dagger\gamma^0\gamma^i\psi$$

Normalization: $\bar{u}u = 2m = -\bar{v}v$ (1.30)

$\bar{u} = u^\dagger\gamma^0$, $\bar{v} = v^\dagger\gamma^0$

Dirac's interpretation of states with E < 0:

In ground state (vacuum) all states with $E < 0$ are filled ('Dirac sea'). Due to Pauli principle, no state with $E > 0$ can undergo a transition to a state with $E < 0$.

But: By introducing an amount of energy $\Delta E > 2m$

one can lift one particle from the Dirac sea into an $E > 0$ state, leaving behind a 'hole' in the Dirac sea. This behaves like an antiparticle, with opposite charge and spin, and with positive energy. This hole can be filled in again.

Problem: Does not work for bosons: no Pauli exclusion!

=> this interpretation does not work for K-G equation!

Feynman's interpretation of states with E < 0:

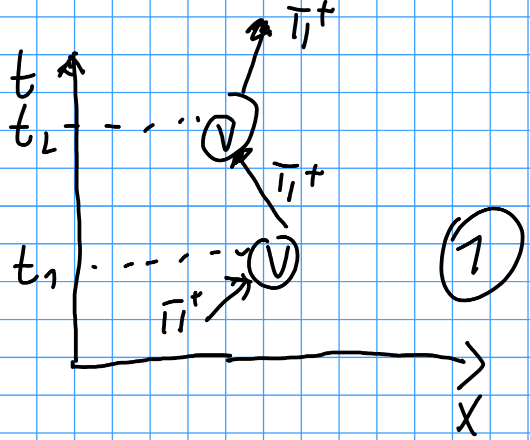
Particle with $E < 0$ moving forward in time = antiparticle with $E > 0$ moving backward in time. (1.30)

Recall: $\psi(x) \hat{=} u(p)\, e^{-ip\cdot x}$ "$E>0$" particle

$= v(p)\, e^{ip\cdot x}$ "$E<0$" antiparticle ($p^0 > 0$)

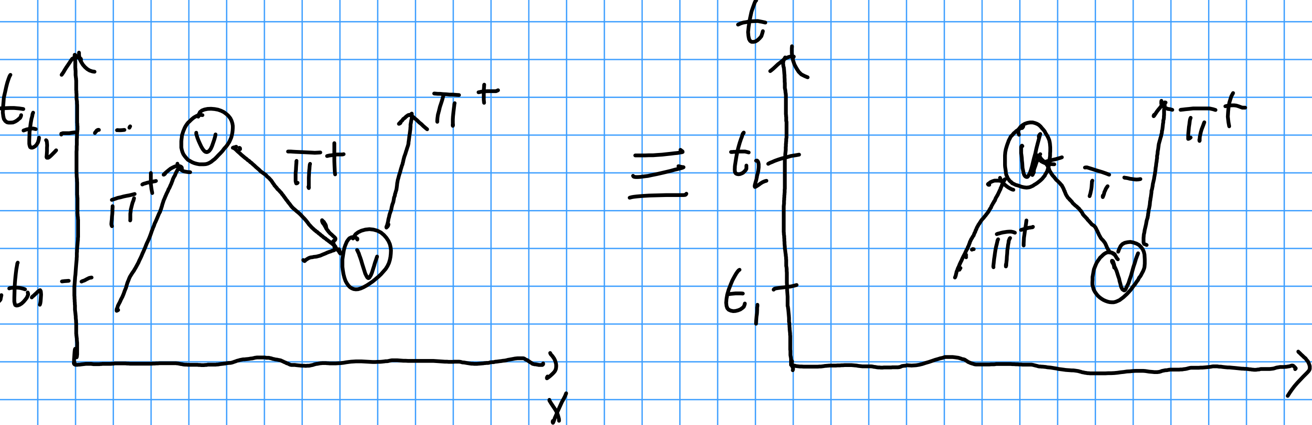
This means: The emission (absorption) of an antiparticle with 4-momentum p^μ is equivalent to the absorption (emission) of a particle w/ 4-mom. $-p_\mu$

E.g. potential scattering of π^+ to 2^{nd} order in pert. theory:



π^+ first scatters at time t_1, then at time t_2

However, in relat. QFT: need to include scattering "backward in time":



The fact that $\rho \equiv \psi^\dagger \psi > 0$ for both "particle" and "antiparticle" becomes problematic: In order to interpret ρ as charge density, need to put some signs "by hand".

This is <u>also</u> true for transition amplitude

$$S_{fi} \equiv \langle \text{final} | \text{initial} \rangle \simeq 1 + A_{fi}$$

$$A\left(X \underset{\text{emission}}{\longrightarrow} X' + \text{particle}(p^\mu, s_z)\right) = A\left(X \underset{\text{absorption}}{+} \text{antipart.}(-p^\mu, -s_z) \to X'\right)$$

$$A(X \to Y + \text{antip.}(-p^\mu, -s_z)) = \overset{\text{boson}}{\underset{\text{fermion}}{\pm}} A(X + \text{particle}(p^\mu, s_z) \to Y) \tag{1.31}$$

X, X', Y : some (many-particle) state.

(Proof: QFT)

Through Feynman's interpretation: can treat many processes using single-particle states only; otherwise would need to include particle - antiparticle pair creation as in fig.(2) above