2) Interacting field theories

Content: a) Lagrangian treatment

b) Examples

c) Perturbation theory and Feynman rules

a) Lagrangian treatment

Basic postulate: action S should be stationary (i.e., invariant) under variations of the dynamical variables:

$$\delta S = 0 \qquad (2.1)$$

In classical mechanics of point particles: dynamical variables are generalized coordinates $q_i(t)$

Is equivalent to Newtonian mechanics $m\ddot{\vec{x}} = \underbrace{\vec{F}}_{\text{force}}(\dot{x}, t)$ (2.2)

Action: $S = \int dt\, \underbrace{L}_{\text{Lagrange fct.}}(q_i(t), \dot{q}_i(t), t)$ (2.3)

Here: are dealing with wave fuctions, which are functions of space and time. (As are the fields in classical electrodynamics.)

$$S = \int d^4x\, \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \qquad (2.4)$$

Lagrange density (Lagrangian) is a functional. It can depend on the fields (wave fct) and their 1st derivative $\partial_\mu \phi_i = \frac{\partial \phi_i}{\partial x^\mu}$

Variation of ϕ_i implies variation of $\partial_\mu \phi_i$ ⇒ variation of S:

$$\delta S = \int d^4x \, \delta \mathcal{L} = \int d^4x \sum_i \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta \phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) \right] \quad (2.5)$$

fields on boundaries are fixed

Functional derivative: computed like ordinary derivative

Since $\delta(\partial_\mu \phi_i) = \partial_\mu \delta \phi_i$: last term in (2.5) can be integrated by parts:

$$\delta S = \int d^4x \sum_i \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta \phi_i - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \delta \phi_i \right] \quad (2.6)$$

$+$ "surface terms"

The surface term is a 3-dimensional integral with one coordinate on the integration boundary. By definition the $\delta \phi_i = 0$ there

⇒ surface terms $= 0$

Remark: Total derivatives of Lagrangian, i.e. surface terms in the action, can (almost) always be neglected: fields are assumed to drop off sufficiently quickly as some coordinate goes to infinity.

(2.6) must hold $\forall \delta \phi_i$: ∞ degrees of freedom

⇒ integrand must vanish: $\boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0} \quad \forall_i \quad (2.7)$

Euler-Lagrange eqs. of motion (e.o.m): one for each field.

In quantum mechanics: Hamilton operator crucial

In classical mechanics: $H = \sum_i p_i \dot{q}_i - L \quad (2.8)$

Canonical momentum variable $p_i \equiv \partial L / \partial \dot{q}_i$ (2.9)

Here: conjugate "momentum" fields $\pi_i(x) \equiv \partial \mathcal{L} / \partial(\dot{\phi}_i)$ (2.10)

$\dot{\phi}_i \equiv \partial \phi_i / \partial t$: time is singled out : manifest Lorentz invariance is lost!

Hamiltonian density $\mathcal{H} = \sum_i \pi_i(x)\, \dot{\phi}_i(x) - \mathcal{L}$ (2.11)

Remarks:

*) S is Lorentz invariant, 4-volume element d^4x is Lorentz invariant

$\Rightarrow \mathcal{L}$ is Lorentz-invariant

*) Hamiltonian density is not Lorentz invariant
=> usually prefers Lagrangian to describe relativistic field theories

*) Action has no units ($\hbar = 1$); $\int d^4x$ has unit $[E]^{-4}$

$\Rightarrow \mathcal{L}$ (and $\mathcal{H}$) must have units $[E]^4$!

*) Lagrangian (and Hamiltonian) must be hermitean (action is real)

Noether's Theorem

Assume $\mathcal{L}$ is invariant under some continuous trafo of the fields:

$$\phi_i(x) \to \phi_i(x) + \varepsilon\, (\Delta\phi)_i(x)$$
$$\mathcal{L} \to \mathcal{L} \qquad (2.12)$$

It will be sufficient to consider infinitesimal trafos, $|\varepsilon| \ll 1$ and constant

$\Delta \mathcal{L} = 0$

$$\Rightarrow \sum_i \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \varepsilon (\Delta\phi)_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu \left(\overset{\partial_\mu \varepsilon = 0}{\varepsilon} (\Delta\phi)_i \right) \right] = 0$$

$$\Rightarrow \sum_i \left\{ \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right)}_{= 0 \ (2.7) \ (\text{e.o.m.})} \varepsilon (\Delta\phi)_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \varepsilon (\Delta\phi)_i \right) \right\} = 0$$

$$\Rightarrow \partial_\mu J^\mu = 0 \ , \text{ with } J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \Delta\phi_i \qquad (2.13)$$

J^μ is a conserved (Noether) current. $[J^\mu] = [\varepsilon^{-1}]$.

In class. mech.: time translation invariance $\Rightarrow$ energy conserv.

spatial translation invariance $\Rightarrow$ linear momentum conserv.

Invariance under rotations $\Rightarrow$ angular mom. conserv.

Here: trafos acting on fields only: "internal symmetries"

Remarks:

*) Have one concerves current for each independent symmetry (independent parameter ε)

*) In many applications (gauge theories): Lagrangian remains invariant even if trafo depends on x. These larger symmetries also permit $\varepsilon \equiv$ const. $\Rightarrow$ proof goes through

*) Can be extended to case where $\mathcal{L}$ transforms like total derivative:

$\mathcal{L} \to \mathcal{L} + \partial_\mu K^\mu$ for some K^μ: leaves e.o.m. unchanged

Noether current then also depends on K

*) Symmetries are central to modern understanding of (particle) physics!