2b) Examples

Let's try to find Lagrangians whose e.o.m. are the Klein-Gordon and Dirac eq., respectively.

Free scalar field (complex)

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 |\phi|^2 \qquad (2.14)$$

Note: ϕ, ϕ^* can be treated as independent, i.e. $\partial\phi^*/\partial\phi \equiv 0$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^* \; ; \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*$$

(2.7) $\Rightarrow \partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0$: is (compl. conj.) of K-G eq. (1.2)

One complex scalar field contains 2 real scalar fields; proper normalization $\phi = \frac{1}{\sqrt{2}}(\phi_r + i\phi_i)$, $\phi_r, \phi_i \in \mathbb{R}$ (2.15)

$$\Rightarrow \mathcal{L} = \frac{1}{2}[\partial_\mu(\phi_r + i\phi_i)][\partial^\mu(\phi_r - i\phi_i)] - \frac{m^2}{2}(\phi_i^2 + \phi_r^2)$$

$$= \frac{1}{2}\left[(\partial_\mu \phi_r)(\partial^\mu \phi_r) + (\partial_\mu \phi_i)(\partial^\mu \phi_i) - m^2(\phi_r^2 + \phi_i^2)\right] \qquad (2.16)$$

$\partial_\mu \phi_r$, $\partial^\mu \phi_r$ are basically same thing:

$$\frac{\partial[(\partial_\mu \phi_r)(\partial^\mu \phi_r)]}{\partial(\partial_\nu \phi_r)} = 2\partial^\nu \phi_r$$

To see this: $\partial_\mu \phi_r \partial^\mu \phi_r = g^{\mu\alpha} \partial_\mu \phi_r \partial_\alpha \phi_r$

$\Rightarrow$

$$\frac{\partial[(\partial_\mu\phi_r)(\partial^\mu\phi_r)]}{\partial(\partial_\nu\phi_r)} = g^{\mu\alpha}[\delta_\mu^\nu\,\partial_\alpha\phi_r + \partial_\mu\phi_r\,\delta_\alpha^\nu]$$

$$= g^{\nu\alpha}\partial_\alpha\phi_r + g^{\mu\nu}\partial_\mu\phi_r = \partial^\nu\phi_r\cdot 2 \;\checkmark\checkmark$$

$\Rightarrow$ e.o.m. from (2.16):

$$\cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\partial_\nu\partial^\nu\phi_r + \cancel{2}\cdot\frac{1}{\cancel{2}}m^2\phi_r = 0$$

$$\partial_\nu\partial^\nu\phi_i + m^2\phi_i = 0$$

} (2.17): 2 independent, decoupled e.o.m.!

Free fermions:

$$\mathcal{L} = \bar\psi\, i\not\partial\psi - m\bar\psi\psi \quad (2.18)$$

$\not A = A_\mu\gamma^\mu$ (2.19), i.e. $\not\partial = \partial_\mu\gamma^\mu$

Note: $\psi, \bar\psi$ are indep. here $\Rightarrow \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)} = 0 \Rightarrow$

$$\frac{\partial\mathcal{L}}{\partial\bar\psi} \overset{!}{=} 0 = (i\not\partial - m)\psi \;:\; \text{Dirac eq. (1.8)}$$

Hermiticity of $\mathcal{L}$:

$$(\bar\psi\psi)^\dagger = \psi^\dagger\bar\psi^\dagger = \psi^\dagger(\psi^\dagger\gamma^0)^\dagger = \psi^\dagger\overset{\gamma^0}{\overset{\shortparallel}{\gamma^{0\dagger}}}\psi = \psi^\dagger\gamma^0\psi = \bar\psi\psi \checkmark\checkmark$$

$$(\bar\psi i\not\partial\psi)^\dagger = -i(\bar\psi\partial_\mu\gamma^\mu\psi)^\dagger = -i(\partial_\mu\psi)^\dagger\underbrace{\wedge}_{\gamma^0\gamma^0=\mathbb{1}}\gamma^{\mu\dagger}\underbrace{\gamma^0\psi}_{\bar\psi^\dagger}$$

$$= -i(\partial_\mu\bar\psi)\underbrace{\gamma^0\gamma^{\mu\dagger}\gamma^0}_{\gamma^\mu}\psi = -i\partial_\mu(\cancel{\bar\psi\gamma^\mu\psi}) + i\bar\psi\partial_\mu\gamma^\mu\psi$$

total derivative! ignore!

Have used: $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ (2.19) (from (1.10), (1.17))

So far, only recovered known free-field equations! Now let us try to construct interaction terms in the Lagrangian.

These involve three or more field factors in one term.

Recall: $[\mathcal{L}] = [E]^4$; $[\partial_\mu] = [E]$

(2.14) $\Rightarrow$ $[\phi] = [E]$: true for all bosonic fields

(2.18) $\Rightarrow$ $[\psi] = [E]^{3/2}$: true for all fermionic fields (2.20)

An interaction theory of one real scalar ϕ and one Dirac fermion ψ could therefore have the following terms in the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \bar{\psi}i\partial\!\!\!/\psi - m_\psi\bar{\psi}\psi - \frac{1}{6}A\phi^3 + \frac{\lambda}{4!}\phi^4 - \kappa\,\phi\bar{\psi}\psi + \cdots \quad (2.21)$$

$[A] = [E]$; λ, κ are dimensionless

Terms "$+\cdots$" have energy dimension > 4 from fields (e.g. $\phi^5, \phi^6, \phi^2\partial_\mu\phi\partial^\mu\phi, (\bar{\psi}\psi)^2, \ldots$) $\Rightarrow$ coefficients must have <u>negative</u> powers of $[E]$ as units. Theories containing such terms generally have bad quantum / high energy behavior (are not renormalizable, not unitary at high E) $\Rightarrow$ <u>not</u> considered for fundamental theories; can be useful as effective (low-E) theory.

Rule of thumb for fundamental theories in 4 space-time dimensions: Lagrangian should only contain terms whose coefficients are dimensionless ($\varkappa, \lambda$), or have positive power of energy as dimension (masses, A). (2.22)

Equations of motion from (2.21):

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\Rightarrow \partial_\mu \partial^\mu \phi = -m_\phi^2 \phi - \frac{1}{2} A \phi^2 - \frac{\lambda}{6} \phi^3 - \varkappa \bar{\psi} \psi \quad (2.23)$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \Rightarrow i \not{\partial} \psi = m_\psi \psi + \varkappa \phi \psi \quad (2.24)$$

These are coupled, nonlinear e.o.m.: no general exact solution known => rely on perturbation theory!

Simple limiting cases: $\lambda = \varkappa = 0$: "ϕ^3 theory"

$A = \varkappa = 0$: "ϕ^4 theory"

$A = \lambda = 0$: "Yukawa theory"

Still cannot be solved exactly! Have applications in the Higgs sector of the Standard Model.