2c) Perturbation theory, Feynman rules

Start with scalar field theory.

To make contact with non-relativistic scattering theory, write e.o.m.

$$(\partial_\mu \partial^\mu + m_\phi^2)\phi = -V_{KG}\,\phi \qquad (2.25)$$

The transition matrix element ⟨final | initial⟩ is in 1st order pert. th. is the same as in non-relat. scattering theory (justification: QFT):

$$A_{fi}^{(1)} = -i\int d^4x\, \phi_f^* V_{KG}\, \phi_i \qquad (2.26)$$

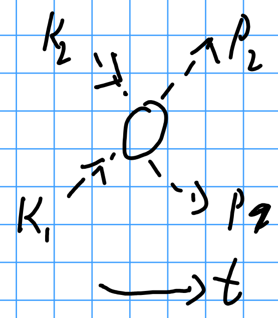
This holds for scattering on "external" potential V_{KG}; in our case, V_{KG} is itself made from fields!

i) ϕ^4 theory (simplest case)

$$(2.23) \Rightarrow (\partial_\mu \partial^\mu + m_\phi^2)\phi = -\frac{\lambda}{6}\phi^3 \qquad (2.27)$$

Want to describe "$2 \to 2$" scattering:

$$\phi(k_1) + \phi(k_2) \to \phi(p_1) + \phi(p_2)$$



Let's assume (2.27) is used to describe evolution of $\phi(k_1)$.
The ϕ^3 factor must then describe $\phi(k_2)$, $\phi(p_1)$, $\phi(p_2)$

There are 3! = 6 ways to distribute these 3 fields over the ϕ^3 factor ⇒ Explains normalization in (2.21)

$$A_{fi}^{(1,\phi^4)} = -i\lambda \int d^4x \, \phi_{k_1}(x)\,\phi_{k_2}(x)\,\phi^*_{p_1}(x)\,\phi^*_{p_2}(x) \qquad (2.28)$$

Perturbative expansion is an expansion in powers of the coupling constant! To 1st order, can use free fields on r.h.s. of (2.28):

$$\phi_p(x) = N(p)\, e^{-ip\cdot x} \qquad (1.6)$$

$$\Rightarrow A_{fi}^{(1,\phi^4)} = -i\lambda\, N(k_1)\,N(k_2)\,N^*(p_1)\,N^*(p_2) \int d^4x\, e^{i(p_1+p_2-k_1-k_2)\cdot x}$$

$$= -i\lambda\, N(k_1) \cdots N^*(p_2)\cdot(2\pi)^4\,\delta^{(4)}(k_1+k_2-p_1-p_2) \qquad (2.29)$$

δ-fct enforces 4-mom. conservation: $k_1+k_2 = p_1+p_2$

Note: up to "trivial" factors (normalization, 4-mom. conserv.) the transition amplitude is just given by the coupling!

Rate of transitions initial --> final $\sim |A_{fi}|^2$; more precisely: want to calculate the cross section!

Normalization: Probability density (1.7): $\rho = 2|N|^2E$.

Take norm.: 2E particles in volume V, i.e. $\int_{Vol.} \rho(\vec{x})\,d^3x = 2E$

$$\Rightarrow N = \frac{1}{\sqrt{V}} \qquad (2.30)$$

Note: V-factors drop out in the end ⇒ $V \to \infty$ ok.

Transition rate per time and volume: $P_{fi} = \frac{|A_{fi}|^2}{VT}$ (2.31)

T: duration of interaction.

Write $A_{fi} = -i\,(2\pi)^4 \prod_i N_i^{(*)} \cdot \delta^{(4)}(\ldots) \cdot F$ (2.32)

F: reduced (Feynman) matrix element; in ϕ^4 theory: $F = \lambda$

$$\Rightarrow P_{fi} = (2\pi)^4\, \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \cdot \prod_i |N_i|^2 \cdot |F|^2$$

"Proof": $$\lim_{V,T\to\infty} \frac{[(2\pi)^4\, \delta^{(4)}(k_1 + k_2 - p_1 - p_2)]^2}{VT}$$

$$= \frac{\int d^4x\, d^4x'\, e^{-ix\cdot(k_1+k_2-p_1-p_2)}\, e^{-ix'\cdot(k_1+k_2-p_1-p_2)}}{\int d^4x \cdot 1}$$

$$= \frac{\int d^4x\, e^{-ix\cdot(k_1+k_2-p_1-p_2)}}{\int d^4x} \cdot (2\pi)^4 \cdot \delta^{(4)}(k_1+k_2-p_1-p_2)$$

$$= (2\pi)^4\, \delta^{(4)}(k_1+k_2-p_1-p_2) \cdot \frac{\int d^4\cancel{x\, e}^{-ix\cdot 0}}{\cancel{\int d^4x}} \qquad \checkmark\checkmark$$

To form a quantity which is independent of experimental set-up, i.e. of the flux of "incident" and "target" particles: have to divide by these factors

*) Flux of "beam" particles on a stationary target (in rest frame of "particle 2") = number of particles reaching target per unit area and unit time: $\mathrm{flux}_1 = |\vec{v}| \cdot \frac{2E_1}{V}$ ↖ 'density' of particle 1

relative velocity between beam & target

*) "Density" of particle 2 $= \frac{2E_2}{V}$ $\left(= \frac{2m_2}{V}\right.$, in rest frame$)$

Finally, need to multiply with final state phase space volume element. For standard normalization (2E particles in V), this is for a 2-particle final state:

$$\frac{V}{(2\pi)^3}\frac{d^3p_1}{2E_1'}\frac{V}{(2\pi)^3}\frac{d^3p_2}{2E_2'} \qquad p_1=(E_1', \vec{p}_1);\ p_2=(E_2', \vec{p}_2)$$

=> differential cross section:

$$d\sigma = P_{fi}\cdot\frac{V}{2E_1|\vec{v}|}\cdot\frac{V}{2E_2}\cdot\frac{V}{(2\pi)^3}\frac{d^3p_1}{2E_1'}\frac{V}{(2\pi)^3}\frac{d^3p_2}{2E_2'}$$

$$\underset{(2.33,\ 2.30)}{=} \frac{1}{4E_1E_2|\vec{v}|}(2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)\cdot\frac{1}{(2\pi)^6}\frac{d^3p_1}{2E_1'}\frac{d^3p_2}{2E_2'}\cdot|F|^2 \qquad (2.35)$$

Note:

*) V has dropped out, as advertized

*) Expression is Lorentz invariant!

$\delta^{(4)}$: obvious ; $\frac{d^3p_i}{2E_i'}$: homework

Flux factor: in rest frame of particle 2:

$k_1=(E_1, \vec{k}_1)$; $k_2=(m_2, \vec{0})$

$\vec{k} = E\cdot(0,0,v)$: in +z direction

Mandelstam-s $= (k_1+k_2)^2 = k_1^2+k_2^2+2k_1\cdot k_2 = m_1^2+m_2^2+2E_1m_2$

$$\Rightarrow s = m_1^2 + m_2^2 + \frac{2m_1}{\sqrt{1-v^2}} \cdot m_2$$

$$\Rightarrow \sqrt{1-v^2} = \frac{2m_1m_2}{s-m_1^2-m_2^2} \Rightarrow v^2 = 1 - \left(\frac{2m_1m_2}{s-m_1^2-m_2^2}\right)^2$$

$$\Rightarrow 4E_1E_2v = 4E_1m_2v = 2\underbrace{(s-m_1^2-m_2^2)}_{2E_1m_2} \cdot \sqrt{1-\left(\frac{2m_1m_2}{s-m_1^2-m_2^2}\right)^2}$$

$$= 2\sqrt{(s-m_1^2-m_2^2)^2 - 4m_1^2m_2^2}$$

$$= 2\sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]} \equiv 2\lambda^{1/2}(s, m_1^2, m_2^2) \quad (2.36)$$

s, m_1^2, m_2^2 are Lorentz invariants $\Rightarrow$ Flux is L.- invariant

Reduced amplitude F is Lorentz invariant

Limiting cases:

-) $m_1 = m_2 \equiv m$ (e.g. e^+e^-, pp, $p\bar{p}$ scattering):

$$4E_1E_2|\vec{v}| = 2s\underbrace{\sqrt{1-4m^2/s}}_{\text{velocity of particle in cms frame}} \quad (2.37)$$

-) $s >> m_1^2, m_2^2$: $4E_1E_2v = 2s$ (2.38)

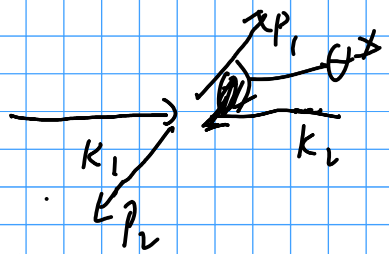
*) Generalization for 2→n scattering: (2.39)

$$d\sigma = \frac{1}{2\lambda^{1/2}(s, m_1^2, m_2^2)} \cdot (2\pi)^4 \delta^{(4)}\left(k_1 + k_2 - \sum_{i=1}^{n} p_i\right) \cdot (2\pi)^{-3n} \cdot \prod_{i=1}^{n} \left(\frac{d^3p_i}{2E_i'}\right) \cdot |F|^2$$

For experimentally useful quantity: need to integrate over sufficiently many final state momenta (components) to get rid of the delta-functions. For 2 --> 2 scattering:

*) In cms frame:

$$\frac{d\sigma}{d\Omega^*} \equiv \frac{d^2\sigma}{\underset{\nearrow\ \text{azimuthal}}{d\phi^*}\ \underset{\hookrightarrow\ \text{polar angle}}{d\cos\theta^*}} = \frac{\lambda^{1/2}(s,m_3^2,m_4^2)}{64\pi^2 s\, \lambda^{1/2}(s,m_1^2,m_2^2)} \cdot |F|^2 \qquad (2.40)$$



θ^*: scattering angle

$$\vec p_1^{\,*} = |\vec p_1^{\,*}| \cdot (\sin\theta^*\cos\phi^*, \sin\theta^*\sin\phi^*, \cos\theta^*) \qquad (2.41)$$

2 → 2 scattering cross sections have nontrivial ϕ^* dependence only if initial particle are transversely polarized: Need to break symmetry under rotations around beam axis!

*) Using Lorentz invariants: Mandelstam t $= (k_1 - p_1)^2 = (k_2 - p_2)^2$

$$\frac{d\sigma(2\to 2)}{dt} = \frac{1}{16\pi\, \lambda(s,m_1^2,m_2^2)} \cdot |F|^2 \qquad (2.42)$$

<u>if</u> $|F|^2$ has no ϕ^* dependence $(\Rightarrow \int d\phi^* = 2\pi)$

*) If final state particles are identical: not allowed to integrate over entire phase space, or divide by a symmetry factor. Reason:

$(\underset{\uparrow\ \text{part.1}}{\vec p_1}, \underset{\uparrow\ \text{part.2}}{\vec p_2})$ and $(\vec p_2, \vec p_1)$ cannot be distinguished!

For n identical particles in final state: need factor $\frac{1}{n!}$ (2.43)

*) $1 \to n$ transitions (particle decay): Very similar reasoning, but normalization ("flux") is now simply $\frac{1}{2E_1}$ $= \frac{1}{2m}$ in rest frame of decaying particle:

$$d\Gamma(1\to n) = \frac{1}{2m} (2\pi)^4 \delta^{(4)}\left(K - \sum_{i=1}^{n} p_i\right) \cdot (2\pi)^{-3n} \prod_{i=1}^{n} \left(\frac{d^3 p_i}{2E_i}\right) \cdot |F|^2 \qquad (2.44)$$

(In rest frame, *not* Lorentz invariant.)