Yukawa theory

Two new ingredients: spin degree of freedom, and Fermi-Dirac statistics for fermions.

Define interaction "potential" V_D through (c.f. 2.25)

$$(i\partial\!\!\!/ - m_\Psi)\Psi \equiv -V_D\,\Psi \quad (2.60) \quad \Rightarrow \quad V_D = -\chi\phi \quad (2.61)$$

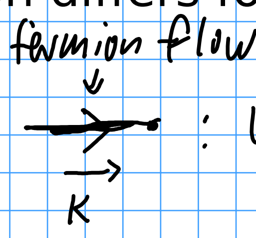
To 1^{st} order in χ we can form one transition amplitude:

$$A^{(1)}_{fi} = i\chi \int d^4x\, \underbrace{\bar{\Psi}(k_3,s_3,x)}_{\text{"outgoing"}}\, \phi(k_2,x)\, \underbrace{\Psi(k_1,s_1,x)}_{\text{"incoming"}} \quad (2.62)$$

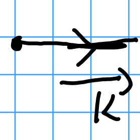
Recall Feynman interpretation:
outgoing antiparticle with momentum k = incoming particle with momentum -k

$$\Psi(\vec{x},t) = u(p)\,\underbrace{e^{-ip\cdot x}}_{\text{incoming}} + v(p)\,\underbrace{e^{ip\cdot x}}_{\text{outgoing}} \quad (1.20)$$

Fermionic wave function differs for particle and antiparticle:

Incoming particle: 

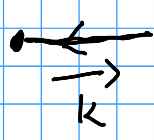
: $u(k)$, with $(k\!\!\!/ - m_\Psi)\,u(k) = 0$ (2.63a)

Outgoing particle: 

: $\bar{u}(k)$, with $\bar{u}(k)(k\!\!\!/ - m_\Psi) = 0$ (2.63b)

Incoming antiparticle: 

: $\bar{v}(k)$, with $\bar{v}(k)(k\!\!\!/ + m_\Psi) = 0$ (2.63c)

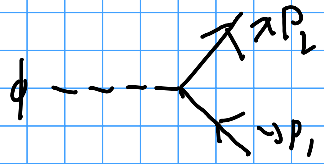
Outgoing antiparticle: 

: $v(k)$, with $(k\!\!\!/ + m_\Psi)\,v(k) = 0$ (2.63d)

Only physical process that can be described by (2.62): decay

$$\phi(K \equiv K_2) \to \bar{\psi}(p_1 \equiv k_3) + \psi(p_2 \equiv K_1) \quad , \text{if } m_\phi > 2 m_\psi$$

Example: $H \to \tau^+ \tau^-$ in SM!



$$A_{fi}(\phi \to \bar{\psi}\psi) = i\chi \int d^4x \, \bar{u}(p_2, s_2) \, v(p_1, s_1) \, e^{-ix\cdot(k - p_2 - p_1)}$$

$$= i\chi \, (2\pi)^4 \, \delta^{(4)}(k - p_1 - p_2) \, \bar{u}(p_2, s_2) \, v(p_1, s_1) \quad (2.64)$$

Note:

*) Have to go through Feynman diagram against the flow of the Dirac arrow in order to get spinors in the order required for proper matrix product: get product of row matrix with column matrix, giving a "1 x 1 matrix", i.e. a complex number.

*) 4-momentum conservation works as in purely theories

*) Have omitted V factors: cancel out in the end again

Often: want to sum (or average, in the initial state) over all spins (i.e., are not measuring polarization, or have unpolarized initial state). This leads to a trick when calculating squared matrix elements:

$$\overline{|F|^2} \equiv \sum_{\text{spins}} |F|^2 = \sum_{\substack{s_1, s_2 = \pm \\ a,b = 1,\dots,4}} \chi^2 \, \bar{u}(p_2, s_2)_a \, v(p_1, s_1)_a \left(\bar{u}(p_2, s_2)_b \, v(p_1, s_1)_b \right)^\dagger$$

$$\Rightarrow |\bar{F}|^2 = \sum_{\substack{s_1,s_2\\a,b}} \varkappa^2\, \bar{u}(p_2,s_2)_a V(p_1,s_1)_a V^\dagger(p_1,s_1)_b\, \bar{u}^\dagger(p_2,s_2)_b$$

$$= \varkappa^2 \sum_{\substack{s_1,s_2\\a,b,c=1,\dots,4}} \bar{u}(p_2,s_2)_a V(p_1,s_1)_a V^\dagger(p_1,s_1)_b \left[u^\dagger(p_2,s_2)_c\, \gamma^0_{cb}\right]^\dagger$$

$$= \varkappa^2 \sum_{\substack{s_1,s_2\\a,b,c}} \bar{u}(p_2,s_2)_a V(p_1,s_1)_a V^\dagger(p_1,s_1)_b\, \gamma^0_{bc}\, u(p_2,s_2)_c$$

$$= \varkappa^2 \sum_{\substack{s_1,s_2\\a,c}} \bar{u}(p_2,s_2)_a V(p_1,s_1)_a\, \bar{V}(p_1,s_1)_c\, u(p_2,s_2)_c$$

$$= \varkappa^2 \sum_{a,c=1}^{4} (\not{p}_1 - m_\psi)_{ac}\, (\not{p}_2 + m_\psi)_{ca} = \varkappa^2\, \mathrm{tr}\left[(\not{p}_1 - m_\psi)(\not{p}_2 + m_\psi)\right]$$

Squared, spin-summed amplitude leads to a <u>trace</u> over products of γ matrices!

Used (proof: HW)

$$\sum_s u(p,s)\,\bar{u}(p,s) = \not{p} + m_\psi$$
$$\sum_s V(p,s)\,\bar{V}(p,s) = \not{p} - m_\psi \qquad (2.65)$$

More generally: Let Γ be some product of γ matrices:

$$F = \bar{u}(p_2,s_2)\,\Gamma\, V(p_1,s_1) \Rightarrow F^* = \bar{V}(p_1,s_1)\,\bar{\Gamma}\, u(p_2,s_2) \qquad (2.66)$$

mass m_2 mass m_1 with $\bar{\Gamma} \equiv \gamma^0 \Gamma^\dagger \gamma^0$ (2.67)

$$\underline{\text{Proof}}: F^* = F^\dagger = V^\dagger(p_1,s_1)\,\Gamma^\dagger\, \underbrace{\bar{u}^\dagger(p_2,s_2)}_{=\gamma^0 u(p_2,s_2)}$$

$$= V^\dagger(p_1,s_1)\gamma^0\gamma^0\Gamma^\dagger\gamma^0 u(p_2,s_2) = \bar{V}(p_1,s_1)\,\bar{\Gamma}\, u(p_2,s_2) \quad \checkmark\checkmark$$

Also works f. other spinor combinations

Hence: $\sum_{s_1,s_2} |F|^2 = \mathrm{tr}[\Gamma(\not{p}_1 - m_\psi)\bar{\Gamma}(\not{p}_2 + m_\psi)]$ (2.68)

To compute $|F|^2$ for $\phi \to \bar{\psi}\psi$, need

$\mathrm{tr}\,\gamma_\mu = 0$ (2.69) : by inspection

$$\mathrm{tr}(\gamma_\nu\gamma_\mu) \underset{\mathrm{tr}(AB)=\mathrm{tr}(BA)}{=} \tfrac{1}{2}\mathrm{tr}[\gamma_\nu\gamma_\mu + \gamma_\mu\gamma_\nu] \underset{(1.9)}{=} g_{\mu\nu}\cdot\mathrm{tr}\,\mathbb{1}_{4\times4} = 4g_{\mu\nu} \quad (2.70)$$

$$\Rightarrow \mathrm{tr}(\not{a}\not{b}) = a^\mu b^\nu \mathrm{tr}(\gamma_\mu\gamma_\nu) \underset{(2.70)}{=} 4g_{\mu\nu}a^\mu b^\nu = 4a\cdot b \quad (2.71)$$

$$\Rightarrow |F(\phi\to\bar{\psi}\psi)|^2 = \lambda^2\,\mathrm{tr}[(\not{p}_1 - m_\psi)(\not{p}_2 + m_\psi)]$$
$$= \lambda^2[\mathrm{tr}(\not{p}_1\not{p}_2) - m_\psi\mathrm{tr}(\not{p}_2) + m_\psi\mathrm{tr}(\not{p}_1) - m_\psi^2\mathrm{tr}\,\mathbb{1}_{4\times4}]$$
$$\underset{(2.69,\,2.71)}{=} 4\lambda^2(p_1\cdot p_2 - m_\psi^2) \quad (2.72)$$

From $\delta^{(4)}$-fct: $k^2 = (p_1+p_2)^2 \Rightarrow m_\phi^2 = 2m_\psi^2 + 2p_1\cdot p_2$

$\Rightarrow 2p_1\cdot p_2 = m_\phi^2 - 2m_\psi^2$ (2.73)

$$\Rightarrow |F|^2 = \lambda^2\cdot(2m_\phi^2 - 8m_\psi^2) = 2\lambda^2 m_\phi^2\left(1 - \frac{4m_\psi^2}{m_\phi^2}\right) \equiv 2\lambda^2 m_\phi^2\beta^2$$

Here: $\beta \equiv \sqrt{1 - \frac{4m_\psi^2}{m_\phi^2}}$: velocity of ψ (or $\bar{\psi}$) in rest frame of ϕ.

Note: $\beta \to 0$ as $m_\psi \to \frac{1}{2}m_\phi$, i.e. at kinematic threshold.

Occurence of β^2 in amplitude indicates that final state must have orbital angular momentum, $\ell = 1$. Reason: Parity!

*) ϕ is scalar: $\phi \xrightarrow[P]{} \phi$

*) Intrinsic Parity of fermion-antifermion pair is -1

(HW: $u \xrightarrow[P]{} +u$, $v \xrightarrow[P]{} -v$ $\Rightarrow u\bar{v} \xrightarrow[P]{} -u\bar{v}$!)

*) Coupling preserves parity: $\phi\bar{\psi}\psi$ behaves like $m_\psi \bar{\psi}\psi$ under parity

Initial state: $P = +$

Final state: $P = (-1)^{1+\ell}$ (2.75) $\Rightarrow$ need ℓ = odd: 1, 3, 5, ...

Also have to conserve total angular mom.:

$$\vec{J} = \vec{L} + \underbrace{\vec{S}}_{\hookrightarrow 0, \text{ or } 1} \stackrel{!}{=} 0 \quad \Rightarrow \quad \ell = 1 \text{ is only possibility} \Rightarrow |\bar{F}|^2 \sim \beta^{2\ell} = \beta^2$$