Decay width

$$(2.44):\ d\Gamma(\phi\to\bar\psi\psi) = \frac{1}{2m_\phi}(2\pi)^4\delta^{(4)}(k-p_1-p_2)\cdot\frac{d^3p_1}{(2\pi)^3 2E_1'}\,\frac{d^3p_2}{(2\pi)^3 2E_2'}\cdot\left|\bar{F}(\phi\to\bar\psi\psi)\right|^2$$

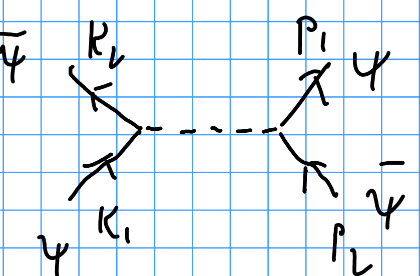
$$\underset{(2.40)}{\Rightarrow}\ \frac{d\Gamma}{d\phi^*\,d\cos\theta^*} = \frac{1}{64\pi^2 \underset{s=m_\phi^2}{m_\phi^3}}\cdot\underbrace{\lambda^{\frac{1}{2}}(m_\phi^2, m_\psi^2, m_\psi^2)}_{m_\phi^2\cdot\beta}\cdot|\bar F|^2$$

$$\underset{(2.73)}{=}\ \frac{\beta}{64\pi^2 m_\phi}\cdot 2\chi^2 m_\phi^2\beta^2 = \frac{\chi^2 m_\phi\beta^3}{32\pi^2}$$

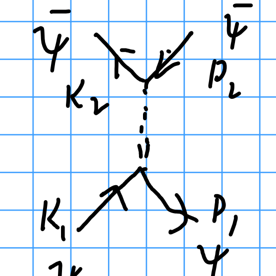
$$\Rightarrow\ \boxed{\Gamma_{tot}(\phi\to\bar\psi\psi) = \frac{\chi^2 m_\phi\beta^3}{8\pi}} \qquad (2.76)$$

To $O(\chi^2)$: several new processes are possible.

i) $\psi(k_1)\,\bar\psi(k_2)\to\psi(p_1)\,\bar\psi(p_2)$

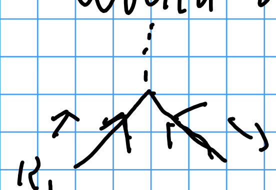


s-channel



t-channel

no u-channel diagram: would need



: "clashing Dirac arrows": forbidden (if $\psi\neq\bar\psi$)

Recall: ψ contains incoming particle (i.p.) + outgoing antipart. (o.a.)

$\bar\psi$ contains outgoing particle (o.p.) + incoming antipart. (i.a.)

$\bar\psi\psi$ contains: (i.p. ⊗ o.p.) + (i.p. ⊗ i.a) + (o.a. ⊗ o.p.) + (o.a. ⊗ i.a.)

lower t-ch. vert. — left s-ch. vert. — right s-ch. vert. — upper t-ch. vert.

$$F(\psi\bar\psi\to\psi\bar\psi) = \bar v(k_2,s_2)(-i\lambda)\,u(k_1,s_1)\,\frac{i}{(k_1+k_2)^2-m_\phi^2}\cdot \bar u(p_1,s_1')(-i\lambda)\,v(p_2,s_2')$$
$$-\bar u(p_1,s_1')(-i\lambda)\,u(k_1,s_1)\,\frac{i}{(k_1-p_1)^2-m_\phi^2}\cdot \bar v(k_2,s_2)(-i\lambda)\,v(p_2,s_2')$$

Note the relative minus sign: Occurs whenever "fermion lines have to be swapped":

s-channel: k_1 couples k_2, p_1 couples to p_2

t-channel: k_1 couples to p_1, k_2 couples to p_2

Basic origin: Fermi-Dirac statistics! Ensures that final state is anitsymmetric under exchange of particles (for identical fermions), i.e. F changes sign when momenta are swapped.
Fully consistent treatment requires QFT ("Wick's theorem")

Note:

*) Overall sign of F is not well defined. However, the relative sign between different contributions does matter!

*) $\overline{|F|^2}$ now usually defined by averaging over initial spin:

$$\overline{|F|^2} = \frac{1}{4} \sum_{\substack{s_1, s_2 = \pm \\ s_1', s_2' = \pm}} |F|^2 \qquad (2.74)$$

*) squared terms give product of 2 short traces, crossed term gives one long trace

$\mathbb{C}$-number

$$F^* = i\kappa^2 \left[\bar{u}(k_1, s_1)\, v(k_2, s_2)\right] \left[\bar{v}(p_2, s_2')\, u(p_1, s_1')\right] \cdot \frac{1}{s - m_\phi^2}$$

$$- i\kappa^2 \left[\bar{u}(k_1, s_1)\, u(p_1, s_1')\right] \left[\bar{v}(p_2, s_2')\, v(k_2, s_2)\right] \cdot \frac{1}{t - m_\phi^2}$$

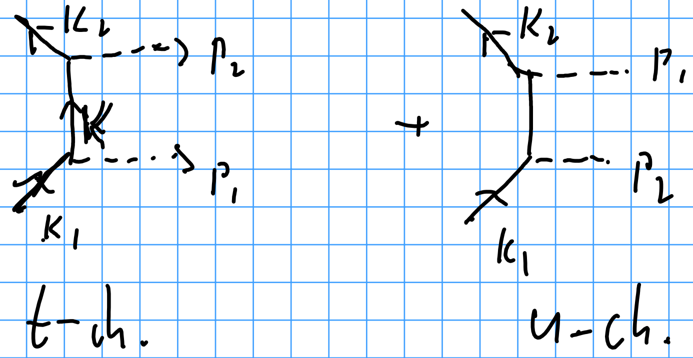
$$\Rightarrow \overline{|F|^2} = \frac{1}{4}\kappa^4 \cdot \Big\{ \frac{1}{(s - m_\phi^2)^2} \cdot \mathrm{tr}\left[(\not{k}_1 + m_\psi)(\not{k}_2 - m_\psi)\right] \cdot \mathrm{tr}\left[(\not{p}_2 - m_\psi)(\not{p}_1 + m_\psi)\right] \qquad (2.76)$$

$$+ \frac{1}{(t - m_\phi^2)^2} \cdot \mathrm{tr}\left[(\not{k}_1 + m_\psi)(\not{p}_1 + m_\psi)\right] \cdot \mathrm{tr}\left[(\not{p}_2 - m_\psi)(\not{k}_2 - m_\psi)\right]$$

$$- \frac{2}{(s - m_\phi^2)(t - m_\phi^2)} \cdot \mathrm{Re}\left\{\mathrm{tr}\left[(\not{k}_1 + m_\psi)(\not{p}_1 + m_\psi)(\not{p}_2 - m_\psi)(\not{k}_2 - m_\psi)\right]\right\} \Big\}$$

ii) To $\mathcal{O}(\kappa^2)$, can also describe $\psi\bar{\psi} \to \phi\phi$

$$\psi(k_1, s_1)\, \bar{\psi}(k_2, s_2) \to \phi(p_1)\, \phi(p_2)$$



Note: Diagrams differ by swapping bosons => relative plus sign!
(Bose-Einstein statistics.)

For internal fermion line: need expression for fermion propagator!
Proceed as for scalar propagator; e.g. for t-channel:

$$A_{fi}^{(t\text{-ch.})} = -i\kappa \int \bar{\psi}_K(x)\, \psi_{k_1}(x)\, \phi^*_{p_1}(x)\, d^4x \qquad (2.77)$$

ψ_K from e.o.m. (2.24)

$$(i\partial\!\!\!/ - m_\psi)\psi_K(x) = \underbrace{\kappa\, \phi_{p_2}(x)\, \psi_{k_2}(x)}_{\text{"source" of virtual particle}} \qquad (2.78)$$

To given order in perturbation theory: use plane wave solutions for all external particles

Ansatz $\quad \psi_K(x) = \underset{\uparrow\ \text{spinor!}}{f(K)}\, e^{-iK\cdot x}$

$v(k_2)$ comes with $e^{+ik_2\cdot x}$

$$\underset{(2.78)}{\Rightarrow} (K\!\!\!/ - m_\psi)\, f(K)\, e^{-iK\cdot x} = \kappa\, v(k_2)\, e^{-ix\cdot(p_2-k_2)}$$

$$\Rightarrow K = p_2 - k_2 = k_1 - p_1\ ; \quad f(K) = \frac{\kappa}{K\!\!\!/ - m_\psi}\, v(k_2) \qquad (2.79)$$

$$\underset{(2.77)}{\Rightarrow} A_{fi}^{(t\text{-ch.})} = -i\kappa^2 \int \bar{v}(k_2, s_2)\, \frac{1}{p\!\!\!/_2 - k\!\!\!/_2 - m_\psi}\, u(k_1, s_1)\, e^{-ix\cdot(k_1+k_2-p_1-p_2)}$$

$$= -i\kappa^2 (2\pi)^4 \delta^{(4)}(k_1+k_2-p_1-p_2)\, \bar{v}(k_2,s_2) \frac{1}{\not{p}_2-\not{k}_2-m_\psi} u(k_1,s_1)$$

$$= (2\pi)^4 \delta^{(4)}(k_1+k_2-p_1-p_2)\, \bar{v}(k_2,s_2)(-i\kappa) \frac{i}{\not{p}_2-\not{k}_2-m_\psi} (-i\kappa)\, u(k_1,s_1) \qquad (2.79)$$

To work out meaning of Dirac matrices in the denominator:

use $\not{a}\not{a} = a_\mu a_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} a_\mu a_\nu (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \overset{(1.9)}{=} a_\mu a_\nu g^{\mu\nu} \cdot \mathbb{1}$

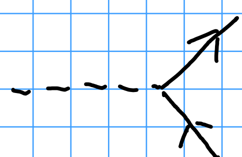
$$= a^2 \cdot \mathbb{1}$$

Hence: $\dfrac{i}{\not{k}-m_\psi} = \dfrac{i(\not{k}+m_\psi)}{(\not{k}-m_\psi)(\not{k}+m_\psi)} = \dfrac{i(\not{k}+m_\psi)}{k^2-m_\psi^2}$: is 4x4 Dirac matrix $\quad (2.80)$

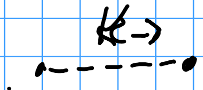
Note: needed $\bar{\psi}_k$, not ψ_k; but: $\bar{\gamma}_\mu \equiv \gamma^0 \gamma_\mu^\dagger \gamma^0 = \gamma_\mu \quad (2.81)$

Feynman rules for Yukawa theory

*) For external fermions: as in eqs.(2.63)

*) For each vertex:  : $-i\kappa \qquad (2.82)$

*) For boson propagator with 4-mom. k flowing through:

 $\dfrac{i}{k^2-m_\phi^2} \qquad (2.83) \equiv (2.57)$

*) For each fermion propagator w/ 4-mom. k flowing in direction of Dirac arrow:

 : $\dfrac{i}{\not{k}-m_\psi} = i\dfrac{\not{k}+m_\psi}{k^2-m_\psi^2} \qquad (2.84)$

Note: relative direction of momentum and fermion number flow matter!

*) To fix the relative sign between different diagrams: mulitply with (-1) for every exchange of fermion lines, either in the initial or final state. (2.85)

Remarks:

*) Propagators have the same form in different theories (for different interactions): is "inverse of the wave equation operator" in momentum space (a.k.a. Green's fct of theory)

*) Amended Feynman rules to include Feynman diagrams with closed loops: analogous to ϕ^3 theory, i.e. get one integral $\int \frac{d^4q}{(2\pi)^4}$ for each propagator. Important new rule:

Extra factor of (-1) for each closed fermion loop. (2.86)

Reason: Fermi-Dirac statistics.