Crash Course homework solutions: Sheet 1

1.1) Since a boost in the opposite direction, with the same speed, should undo the first boost, the obvious guess is:

$$t = \gamma(t' + v x'^1) \; ; \; x^1 = \gamma(x'^1 + v t') \; ; \; x^2 = x'^2 \; ; \; x^3 = x'^3 . \quad (*)$$

The last 2 eqs. are obvious. In order to check the first two eqs., insert them into the first two original eqs:

$$t' = \gamma(t - v x^1) \underset{(*)}{=} \gamma^2 (t' + \cancel{v x'^1} - \cancel{v x'^1} - v^2 t') = \gamma^2 t'(1 - v^2) = t' \checkmark$$

$$x'^1 = \gamma(x^1 - vt) \underset{(*)}{=} \gamma^2 (x'^1 + \cancel{v t'} - \cancel{v t'} - v^2 x'^1) = \gamma^2 x'^1 (1 - v^2) = x'^1 \checkmark\checkmark$$

1.2) We want to show that $\frac{\partial \phi}{\partial x^\mu}$ transforms like x_μ. A general Lorentz trafo on a 4-vector (like x_μ) can be written as

$$x_\nu \to x'_\nu = \Lambda_\nu{}^\mu x_\mu \quad \text{(summation convention on } \mu\text{!)} \quad (1)$$

$$\Rightarrow x^\nu \to \Lambda^\nu{}_\mu x^\mu = x'^\nu$$

$$\frac{\partial \phi}{\partial x^\mu} = \frac{\partial \phi}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\mu} = \frac{\partial \phi}{\partial x'^\nu} \Lambda^\nu{}_\mu \text{ : this is indeed as in eq. (1)!}$$

2) The Pauli matrices are: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow \{\sigma_1, \sigma_1\} = 2\sigma_1^2 = 2 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \cdot \underbrace{\mathbb{1}_{2\times 2}}_{\text{unit matrix}}$$

$$\{\sigma_2, \sigma_2\} = 2\sigma_2^2 = 2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2 \cdot \mathbb{1}_{2\times 2}$$

$$\{\sigma_3, \sigma_3\} = 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\, \mathbb{1}_{2\times 2}$$

$$\{\sigma_1, \sigma_2\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}}_{=0}$$

S1.2

$$\{\sigma_1,\sigma_3\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0$$

$$\{\sigma_2,\sigma_3\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0$$

Altogether: $\{\sigma_i,\sigma_j\} = 2\delta_{ij}\,\mathbb{1}_{2\times2}$ (2)

<u>Hence:</u>

$$\{\gamma^0,\gamma^0\} = 2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = 2\,\mathbb{1}_{4\times4} = 2g^{00}\,\mathbb{1}_{4\times4}$$

$$\{\gamma^i,\gamma^i\} = 2\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = 2\begin{pmatrix} -\sigma^{i2} & 0 \\ 0 & -\sigma^{i2} \end{pmatrix} = -2\,\mathbb{1}_{4\times4}$$

$$= 2g^{ii}\,\mathbb{1}_{4\times4}$$

↑ no sum over i!

$$\{\gamma^0,\gamma^i\} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = 0 \quad ✓$$

$$\{\gamma^i,\gamma^j\} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} + \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}$$

$$= -\begin{pmatrix} \{\sigma^i,\sigma^j\} & 0 \\ 0 & \{\sigma^i,\sigma^j\} \end{pmatrix} \underset{(2)}{=} -2\delta^{ij}\,\mathbb{1}_{4\times4} \quad ✓✓$$

S1.3

3.1) $i\gamma^\mu \partial_\mu \psi - m\psi = 0$ (3) : Original Dirac eq.

Take hermitean conjugate: $-i(\partial_\mu \psi^\dagger)\gamma^{\mu\dagger} - m\psi^\dagger = 0$ (3)

Note: Taking the transpose (and hence also taking the hermitean conjugate) changes the order of a matrix product:

$(A \cdot B)^T = B^T A^T$, where A, B are matrices; in the case at hand, B is a single-column matrix.

Multiply (3) from the right with $-\gamma^0$:

$$i(\partial_\mu \psi^\dagger)\gamma^{\mu\dagger}\gamma^0 + m\psi^\dagger\psi^0 = 0 \qquad (4)$$

Note: The Pauli matrices are hermitean, $\sigma^{i\dagger} = \sigma^i$

$\Rightarrow \gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i \quad (i = 1,2,3)$

$\Rightarrow$ $\gamma^{0\dagger}\gamma^0 = \gamma^0\gamma^0; \quad \gamma^{i\dagger}\gamma^0 = \gamma^0\gamma^{i\dagger}$. Insert in (4)

$\gamma^i\gamma^0 = -\gamma^0\gamma^i$

$\Rightarrow i(\partial_\mu \psi^\dagger\gamma^0)\gamma^\mu + m\psi^\dagger\gamma^0 = 0 \Rightarrow i(\partial_\mu \bar\psi)\gamma^\mu + m\bar\psi = 0$ ✓✓

3.2) Want to show $\partial_\mu(\bar\psi\gamma^\mu\psi) = 0$. By product rule:

$$\partial_\mu(i\bar\psi\gamma^\mu\psi) = (i\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi i\gamma^\mu\partial_\mu\psi$$

$$\underset{\text{Dirac eq.}}{=} (-m\bar\psi)\psi + \bar\psi m\psi = 0 \quad ✓✓$$

3.3) Positive-energy solution: eq. (1.14) in class:

$$u = \sqrt{E+m}\begin{pmatrix}\phi \\ \frac{\vec p\cdot\vec\sigma}{E+m}\phi\end{pmatrix}$$

Here $\phi = \begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$ describe the two spin eigenstates in the rest frame of the particle

Recall: σ matrices are hermitean, as is $\vec{p}$

$$\Rightarrow u^\dagger = \sqrt{E+m}\left(\phi^\dagger, \phi^\dagger \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)$$

Explicitly: $\vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$

$$\bar{u} = u^\dagger \gamma^0 = u^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sqrt{E+m}\left(\phi^\dagger, -\phi^\dagger \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)$$

Hence: $\bar{u}u = (E+m)\left(\phi^\dagger, -\phi^\dagger \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)\begin{pmatrix} \phi \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi \end{pmatrix}$

$$= (E+m)\cdot\left[\underbrace{\phi^\dagger\phi}_{=1} - \phi^\dagger \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi\right]$$

$$\vec{p}\cdot\vec{\sigma}\ \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \begin{pmatrix} p_z^2 + p_x^2 + p_y^2 & 0 \\ 0 & p_x^2 + p_y^2 + p_z^2 \end{pmatrix}$$

$$= \vec{p}^2 \cdot \mathbb{1}$$

$$\Rightarrow \bar{u}u = (E+m)\left[1 - \frac{\vec{p}^2}{(E+m)^2}\right] \overset{\vec{p}^2 = E^2 - m^2}{=} \frac{1}{E+m}\left[(E+m)^2 - E^2 + m^2\right] \quad (\#)$$

$$= \frac{2Em + 2m^2}{E+m} = \frac{2m(E+m)}{E+m} = 2m$$

Negative-energy sol: eq. (1.17) in class

$$\bar{v}v = (E+m)\left(\chi^\dagger \frac{\vec{p}\cdot\vec{\sigma}}{E+m}, \chi^\dagger\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi \\ \chi \end{pmatrix}$$

$$= (E+m)\left[\chi^\dagger \frac{\vec{p}\cdot\vec{\sigma}\ \vec{p}\cdot\vec{\sigma}}{(E+m)^2}\chi - 1\right] \text{ is } (\#)\cdot(-1)!$$

$$= -2m.$$

$\boxed{S1.5}$

3.4) Had: $u = N\begin{pmatrix}\phi \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi\end{pmatrix}$; $v = N\begin{pmatrix}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi \\ \chi\end{pmatrix}$

$\Rightarrow u^* = N\begin{pmatrix}\phi \\ \frac{\vec{p}\cdot\vec{\sigma}^*}{E+m}\phi\end{pmatrix}$ (ϕ is real! $\phi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix} = \chi^{(2)}$, $\phi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix} = \chi^{(1)}$)

$\Rightarrow u^C = i\gamma^2 u^* = \begin{pmatrix}0 & +i\sigma_2 \\ -i\sigma_2 & 0\end{pmatrix} u^* = N\begin{pmatrix}+i\sigma_2\frac{\vec{p}\cdot\vec{\sigma}^*}{E+m}\phi \\ -i\sigma_2\phi\end{pmatrix}$

$\sigma_2 = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} \Rightarrow i\sigma_2 = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} \Rightarrow -i\sigma_2\phi^{(1)} = \begin{pmatrix}0 \\ +1\end{pmatrix}; \ -i\sigma_2\phi^{(2)} = \begin{pmatrix}-1 \\ 0\end{pmatrix}$

$\vec{p}\cdot\vec{\sigma}^* = \begin{pmatrix}p_z & p_x+ip_y \\ p_x-ip_y & -p_z\end{pmatrix} \Rightarrow i\sigma_2\,\vec{p}\cdot\vec{\sigma}^* = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}p_z & p_x+ip_y \\ p_x-ip_y & -p_z\end{pmatrix} = \begin{pmatrix}p_x-ip_y & -p_z \\ -p_z & -p_x-ip_y\end{pmatrix}$

$\Rightarrow i\sigma_2\,\vec{p}\cdot\vec{\sigma}^*\phi_1 = \begin{pmatrix}p_x-ip_y \\ -p_z\end{pmatrix}; \ i\sigma_2\,\vec{p}\cdot\vec{\sigma}^*\phi_2 = \begin{pmatrix}-p_z \\ -p_x-ip_y\end{pmatrix}$

$\Rightarrow u^{(1)C} = N\begin{pmatrix}(p_x-ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1\end{pmatrix}; \ u^{(2)C} = N\begin{pmatrix}-p_z/(E+m) \\ (-p_x-ip_y)/(E+m) \\ -1 \\ 0\end{pmatrix}$

$\vec{p}\cdot\vec{\sigma}\chi_1 = \begin{pmatrix}p_z & p_x-ip_y \\ p_x+ip_y & -p_z\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix} = \begin{pmatrix}p_x-ip_y \\ -p_z\end{pmatrix}; \ \vec{p}\cdot\vec{\sigma}\chi_2 = \begin{pmatrix}-p_z \\ -(p_x+ip_y)\end{pmatrix}$

$\Rightarrow v^{(1)} = N\begin{pmatrix}(p_x-ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1\end{pmatrix} = u^{(1)C}$

$v^{(2)} = N\begin{pmatrix}-p_z/(E+m) \\ -(p_x+ip_y)/(E+m) \\ -1 \\ 0\end{pmatrix} = u^{(2)C}$