2nd sheet

1.1) In general, dx transforms like x itself:

$dx \to dx' = \gamma(dx - \beta dt)$, or its inverse

$dx = \gamma(dx' + \beta dt')$. However, the 3-volume in the primed system is <u>defined</u> at fixed t', i.e. $dt' = 0$

$\Rightarrow dx' = dx/\gamma$ (1) : this is the famous Lorentz contraction

dy, dz are not changed $\Rightarrow d^3x \to d^3x/\gamma$ is <u>not</u> invariant.

1.2) When changing frame, $dt \to \gamma dt$ (time dilation), hence $dx\,dt$ is <u>invariant</u>, and so is the 4-volume element d^4x.

1.3) $p^2 \equiv p_\mu p^\mu$ is invariant, and m^2 is trivially invariant. Any function of invariants is itself invariant, i.e. $\delta(p^2 - m^2)$ is the same in all reference frames (connected by Lorentz trafos)

1.4) $\frac{d^3p}{2E}$ is invariant. One way to see this is to consider

$$\int_0^\infty dE\, \delta(p^2 - m^2) = \int_0^\infty dE\, \delta(E^2 - \vec{p}^2 - m^2)$$

Recall: $\int dx\, \delta(f(x)) = \frac{1}{f'(x_0)}$, with $f(x_0) = 0$. Here, the argument of the δ-fct. vanishes if $E = \sqrt{\vec{p}^2 + m^2}$, hence

$$\int_0^\infty dE\, \delta(p^2 - m^2) = \frac{1}{2\sqrt{\vec{p}^2 + m^2}} = \frac{1}{2E} \quad \text{(for on-shell particles)}$$

Hence $\int \frac{d^3p}{2E} = \int d^4p\, \delta(p^2 - m^2)$, which is clearly invariant (by Ex. 1.3)

S2.2

2.1) ϕ is complex $\Rightarrow$ ϕ, ϕ^* can be considered independent variables (rather than $\mathrm{Re}(\phi)$, $\mathrm{Im}(\phi)$ for example)

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\right) = \partial_\mu \partial^\mu \phi = + \frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi$$

$$\Rightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0 \quad \text{; Klein-Gordon-eq.} \qquad (*)$$

2.2) Since α is a constant, the factors $e^{i\alpha}$ (from ϕ) and $e^{-i\alpha}$ (from ϕ^*) can be pulled in front of the derivative ∂_μ, and then cancel in $\partial_\mu \phi^* \partial^\mu \phi$; $|\phi|^2$ is also obviously invariant
$\Rightarrow$ $\mathcal{L}$ is invariant under $\phi \to e^{i\alpha}\phi$.

2.3) For infinitesimal α: $\phi \to (1+i\alpha)\phi$, $\phi^* \to (1-i\alpha)\phi^*$
In the notation of eq. (2.12) in class:

$$\Delta\phi = i\phi, \quad \Delta\phi^* = -i\phi^*$$

Hence Noether current from eq. (2.13) in class:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\Delta\phi^* \qquad \text{(note: } \phi, \phi^* \text{ are independent!)}$$

$$= (\partial^\mu \phi^*) i\phi + (\partial^\mu \phi)(-i\phi^*) = i\left[(\partial^\mu \phi^*)\phi - \phi^* \partial^\mu \phi\right]$$

2.4)
$$\partial_\mu j^\mu = i\cdot\left[(\partial_\mu \partial^\mu \phi^*)\phi + \cancel{(\partial^\mu \phi^*)(\partial_\mu \phi)} - \cancel{(\partial_\mu \phi^*)\partial^\mu \phi} - \phi^* \partial_\mu \partial^\mu \phi\right]$$

$$\underset{(*)}{=} i\left[(-m^2\phi^*)\phi + \phi^*(m^2\phi)\right] = im^2\left[-|\phi|^2 + |\phi|^2\right] = 0 \checkmark$$

3.1) $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$= -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\, g^{\mu\rho} g^{\nu\sigma}$$

Need $\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)}$: gets 4 contributions!

Kronecker-δ!

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{4} \cdot \Big[(\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})(\partial_\rho A_\sigma - \partial_\sigma A_\rho) + (\partial_\mu A_\nu - \partial_\nu A_\mu)(\delta_{\rho\alpha}\delta_{\beta\sigma} - \delta_{\sigma\alpha}\delta_{\rho\beta}) \Big] g^{\mu\rho} g^{\nu\sigma}$$

$$= -\frac{1}{4} \Big[(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})(\partial^\mu A^\nu - \partial^\nu A^\mu) + (\partial^\rho A^\sigma - \partial^\sigma A^\rho)(\delta_{\rho\alpha}\delta_{\beta\sigma} - \delta_{\sigma\alpha}\delta_{\rho\beta}) \Big]$$

$$= -\frac{1}{4} \Big[\partial^\alpha A^\beta - \partial^\beta A^\alpha - \partial^\beta A^\alpha + \partial^\alpha A^\beta + \partial^\alpha A^\beta - \partial^\beta A^\alpha - \partial^\beta A^\alpha + \partial^\alpha A^\beta \Big] = -(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = -F^{\alpha\beta}$$

$\Rightarrow$ e.o.m.: $\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = \frac{\partial \mathcal{L}}{\partial A_\beta} = 0$ (note: A_β and derivatives $\partial_\alpha A_\beta$ are independent!)

$\Rightarrow \boxed{\partial_\alpha F^{\alpha\beta} = 0}$: Covariant form of free Maxwell eqs.

Note: Since $\mathcal{L}$ is of 2nd order in derivatives, the fields (A^α) and their first derivatives ($\partial^\beta A^\alpha$) are taken as independent variables, just like generalized coordinates q and their time derivatives $\dot{q}$ in ordinary Lagrangian mechanics.

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad ; \quad \partial_\alpha = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$\Rightarrow$ e.o.m. with $\beta = 0$: $\vec{\nabla}\cdot\vec{E} = \operatorname{div}\vec{E} = 0$ ✓ (no matter present!) S2.4

e.o.m. with $\beta = 1$: $\frac{\partial}{\partial t}E_x - \frac{\partial}{\partial y}B_z + \frac{\partial}{\partial z}B_y = 0 \Rightarrow \frac{\partial E_x}{\partial t} = (\operatorname{rot}\vec{B})_x$

The y, z components are analogous.

3.2) Had seen: $\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -F^{\alpha\beta}$

$$\Rightarrow T^{\mu\nu} = \underbrace{-F^{\mu\alpha}\,\partial^\nu A_\alpha}_{\text{not symmetric in } \mu \leftrightarrow \nu} + \underbrace{\tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}}_{\text{symmetric in } \mu\leftrightarrow\nu}$$

$$K^{\alpha\mu\nu} = F^{\mu\alpha}A^\nu \Rightarrow \partial_\alpha K^{\alpha\mu\nu} = \underbrace{\cancel{(\partial_\alpha F^{\mu\alpha})}}_{\to 0 \text{ by e.o.m.}} A^\nu + F^{\mu\alpha}\partial_\alpha A^\nu$$

$$\Rightarrow \hat{T}^{\mu\nu} = -F^{\mu\alpha}(\partial^\nu A_\alpha - \partial_\alpha A^\nu) + \tfrac{1}{4} g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$= \underbrace{-F^{\mu\alpha}F^{\nu}{}_{\alpha}}_{\text{also symmetric!}} + \tfrac{1}{4} g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

1 spatial index: changes sign when subscripts are lowered

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

2 spatial indices: no change of sign

$$\Rightarrow F_{\alpha\beta}F^{\alpha\beta} = 2(|\vec{B}|^2 - |\vec{E}|^2)$$

$$F^{\nu}{}_{\alpha} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

"time" index lowered: no change

spatial index lowered $\Rightarrow$ change of sign

$-F^{\mu\alpha}F^{\nu}{}_{\alpha}$: summation is both time over the 2nd index! Hence the (μ,ν) comp. is the product of the μ^{th} and ν^{th} row of the two matrices

$$\Rightarrow -F^{0\alpha}F^{0}{}_{\alpha} = E_x^2 + E_y^2 + E_z^2 = |\vec{E}|^2$$

$$\Rightarrow T^{00} = |\vec{E}|^2 + \frac{1}{2}\underbrace{g^{00}}_{1}\cdot(|\vec{B}|^2 - |\vec{E}|^2) = \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2) : \text{energy density}$$

$$\left.\begin{aligned} T^{01} &= -F^{0\alpha}F^{1}{}_{\alpha} = E_y B_z - E_z B_y = (\vec{E}\times\vec{B})_x \\ T^{02} &= -F^{0\alpha}F^{2}{}_{\alpha} = -E_x B_z + E_z B_x = (\vec{E}\times\vec{B})_y \\ T^{03} &= -F^{0\alpha}F^{3}{}_{\alpha} = E_x B_y - E_y B_x = (\vec{E}\times\vec{B})_z \end{aligned}\right\} \vec{S} : \text{energy flow}$$