3rd sheet

1a) $s+t+u = (p_A+p_B)^2 + (p_A-p_C)^2 + (p_A-p_D)^2$

$= p_A^2 + p_B^2 + 2p_A\cdot p_B + p_A^2 + p_C^2 - 2p_A\cdot p_C + p_A^2 + p_D^2 - 2p_A\cdot p_D$

$= 3p_A^2 + p_B^2 + p_C^2 + p_D^2 + \underbrace{2p_A\cdot(p_B - p_C - p_D)}_{= -2p_A^2}$

$p_A + p_B = p_C + p_D$ 4-mom. conservation

$\Rightarrow p_B = p_C + p_D - p_A$

$\Rightarrow s+t+u = p_A^2 + p_B^2 + p_C^2 + p_D^2 = m_A^2 + m_B^2 + m_C^2 + m_D^2$ for on-shell (physical) particles.

1b) In CM frame: $\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D = 0 \Rightarrow \vec{p}_D = -\vec{p}_C$

$\Rightarrow s = (p_A+p_B)^2 = (p_C+p_D)^2 = m_C^2 + m_D^2 + 2p_C\cdot p_D$

$= m_C^2 + m_D^2 + 2E_CE_D - 2\vec{p}_C\cdot\vec{p}_D$

$\underset{\vec{p}_D=-\vec{p}_C}{=} m_C^2 + m_D^2 + 2E_CE_D + 2\vec{p}_C^{\,2}$ $\qquad \vec{p}_C^{\,2} = E_C^2 - m_C^2$

$= m_C^2 + m_D^2 + 2E_CE_D + 2E_C^2 - 2m_C^2$ $\qquad \vec{p}_C + \vec{p}_D = 0$

$= m_D^2 - m_C^2 + 2E_C(E_C+E_D)$ $\qquad s = (p_C+p_D)^2 = (E_C+E_D)^2$

$= m_D^2 - m_C^2 + 2E_C\sqrt{s}$

$\Rightarrow E_C = \dfrac{s + m_C^2 - m_D^2}{2\sqrt{s}}$: only depends on s, not on t,u: this is only true in the CM frame!

1c) In rest frame of B:

$$s=(p_A+p_B)^2 = p_A^2+p_B^2+2p_A\cdot p_B = m_A^2+m_B^2+2E_A \underbrace{E_B}_{=m_B} - 2\vec{p}_A\cdot \underbrace{\vec{p}_B}_{=0}$$

$$\Rightarrow \underline{s = 2E_A m_B + m_A^2 + m_B^2} \xrightarrow[E_A \gg m_A, m_B]{} 2E_A m_B$$

This only grows linearly with E_A. In contrast, for two colliding particles, $s=(E_A+E_B)^2$ (in CM frame) grows quadratically. Hence colliders need much lower beam energy to achieve the same s than fixed-target experiments; hence the construction of colliders from the 1960's on.

1d) In CM frame, again:

Had $E_C = \dfrac{s-m_D^2+m_C^2}{2\sqrt{s}} \Rightarrow E_D = \dfrac{s-m_C^2+m_D^2}{2\sqrt{s}}$ $(E_C+E_D=\sqrt{s})$

This fixes $|\vec{p}_C| = \sqrt{E_C^2-m_C^2} = |\vec{p}_D| = \sqrt{E_D^2-m_D^2}$

$$= \left[\frac{(s-m_D^2+m_C^2)^2-4sm_C^2}{4s}\right]^{1/2} = \left[\frac{s^2+m_C^4+m_D^4 - 2m_C^2m_D^2 -2sm_C^2-2sm_D^2}{4s}\right]^{1/2}$$

$$= \frac{\sqrt{s}}{2}\cdot\left[1+\left(\frac{m_C^2}{s}\right)^2+\left(\frac{m_D^2}{s}\right)^2 - 2\frac{m_C^2}{s} - \frac{2m_D^2}{s} - \frac{2m_C^2m_D^2}{s^2}\right]^{1/2}$$

$$\equiv \frac{\sqrt{s}}{2}\cdot \lambda^{1/2}\left(1, \frac{m_C^2}{s}, \frac{m_D^2}{s}\right)$$

Kinematic fct: symmetric in all 3 arguments

$$t=(p_A-p_C)^2 = m_A^2+m_C^2-2p_A\cdot p_C$$

$$= m_A^2+m_C^2-2E_AE_C+2\vec{p}_A\cdot\vec{p}_C$$

$$= m_A^2+m_C^2-2E_AE_C+2|\vec{p}_A||\vec{p}_C|\cos\theta$$

scattering angle

$$E_A = \frac{s + m_A^2 - m_B^2}{2\sqrt{s}} \quad \text{by analogy w/ } E_C \text{ (in CM frame!)}$$

$$\Rightarrow m_A^2 + m_C^2 - 2E_A E_C = \frac{2s m_A^2 + 2s m_C^2 - (s + m_A^2 - m_B^2)(s + m_C^2 - m_D^2)}{2s}$$

$$= \frac{-s^2 - (m_A^2 - m_B^2)(m_C^2 - m_D^2) + s(m_A^2 + m_B^2 + m_C^2 + m_D^2)}{2s}$$

$$\equiv t_0$$

$$\Rightarrow t \in [t_0 - 2|\mathring{p}_A||\mathring{p}_C|], [t_0 + 2|\mathring{p}_A||\mathring{p}_C|]$$

$$\Rightarrow t \in \left[t_0 - \frac{s}{2} \lambda^{1/2}\left(1, \frac{m_A^2}{s}, \frac{m_B^2}{s}\right) \cdot \lambda^{1/2}\left(1, \frac{m_C^2}{s}, \frac{m_D^2}{s}\right),\right.$$
$$\left. t_0 + \frac{s}{2} \lambda^{1/2}\left(1, \frac{m_A^2}{s}, \frac{m_B^2}{s}\right) \cdot \lambda^{1/2}\left(1, \frac{m_C^2}{s}, \frac{m_D^2}{s}\right)\right]$$

Important special case: collision of ultra-relativistic particles, $s \gg m_A^2, m_B^2 \Rightarrow$ can set $m_A = m_B = 0$. Consider in addition $m_C^2 = m_D^2 \equiv m^2$ (pair production of massive particles)

$$\Rightarrow \lambda^{1/2}\left(1, \frac{m_A^2}{s}, \frac{m_B^2}{s}\right) = 1; \quad \lambda^{1/2}\left(1, \frac{m_C^2}{s}, \frac{m_D^2}{s}\right) = \left[1 - \frac{4m^2}{s}\right]^{1/2}$$

$$t_0 = -\frac{s}{2} + m^2$$

$$\Rightarrow t \in \left[-\frac{s}{2} + m^2 - \frac{s}{2}\sqrt{1 - \frac{4m^2}{s}}, \; -\frac{s}{2} + m^2 + \frac{s}{2}\sqrt{1 - \frac{4m^2}{s}}\right]$$

$$\Rightarrow t \in \left[m^2 - \frac{s}{2}\left(1 + \sqrt{1 - \frac{4m^2}{s}}\right), m^2 - \frac{s}{2}\left(1 - \sqrt{1 - \frac{4m^2}{s}}\right)\right]$$

In particular, for $m^2 \to 0$ (i.e. $s \gg m^2$): $t \in [-s, 0]$

<u>Note</u>: s is always positive, t and u are always negative (or zero) here; the range for u is the same as that for t if $m_C^2 = m_D^2$.

2a) 2^{nd} equality in (2):

$$s = (p_1 + p_2)^2 = 2 p_1 \cdot p_2 + m_1^2 + m_2^2$$

$$\Rightarrow p_1 \cdot p_2 = \frac{1}{2}(s - m_1^2 - m_2^2)$$

$$\Rightarrow (p_1 \cdot p_2)^2 - m_1^2 m_2^2 = \frac{1}{4}\left[(s - m_1^2 - m_2^2)^2 - 4 m_1^2 m_2^2\right]$$

$$= \frac{1}{4} d(s, m_1^2, m_2^2)$$

$$\Rightarrow 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = 2\, d^{1/2}(s, m_1^2, m_2^2) \quad \checkmark\checkmark$$

This is expressed in terms of Lorentz invariants (s, $p_1 \cdot p_2$, m_1^2, m_2^2), hence it holds in every (inertial) frame!

In the lab frame $\equiv$ rest frame of particle 2:

$$4 E_1 E_2 |\vec{v}| = 4 |\vec{p}_1| m_2 = 4\sqrt{E_1^2 - m_1^2} \cdot m_2 \quad (*)$$

Since $p_2 = (m_2, \vec{0})$ in this frame: $p_1 \cdot p_2 = E_1 m_2$

$$\Rightarrow (p_1 \cdot p_2)^2 - m_1^2 m_2^2 = E_1^2 m_2^2 - m_1^2 m_2^2 = m_2^2 (E_1^2 - m_1^2)$$

$$\underset{(*)}{\Rightarrow} 4\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]^{1/2} = 4 E_1 E_2 |\vec{v}| \quad \checkmark$$

CM frame: $\vec{p}_1 = -\vec{p}_2 \Rightarrow E_1 \vec{v}_1 = -E_2 \vec{v}_2 \Rightarrow \vec{v}_2 = -\frac{E_1}{E_2}\vec{v}_1$

$$\Rightarrow 4 E_1 E_2 |\vec{v}| \equiv 4 E_1 E_2 |\vec{v}_1 - \vec{v}_2| = 4 E_1 |\vec{v}_1| E_2 \left(1 + \frac{E_1}{E_2}\right)$$

$$= 4 |\vec{p}_1| \underbrace{E_2 \left(1 + \frac{E_1}{E_2}\right)}_{= E_2 + E_1 = \sqrt{s}} = 2\, d^{1/2}(s, m_1^2, m_2^2) \quad \checkmark\checkmark$$

$|\vec{p}_1| \to \frac{1}{2\sqrt{s}} d^{1/2}(s, m_1^2, m_2^2)$: See expression for $|\vec{p}_c|$ in 1d) above

2b) $\sigma = \frac{\sqrt{s}}{2\lambda^{1/2}(s, m_1^2, m_2^2)} (2\pi)^4 \int \frac{d^3 p_3}{2E_3} \frac{d^3 p_4}{2E_4} \frac{1}{(2\pi)^6} |F|^2 \delta^{(4)}(p_1+p_2-p_3-p_4)$ $\boxed{S3.5}$

$$= \frac{1}{8\pi^2 \lambda^{1/2}(s, m_1^2, m_2^2)} \cdot \int \frac{d^3 p_3}{4E_3 E_4} |F|^2 \delta(\underbrace{E_1+E_2}_{\sqrt{s},\ \text{in CM frame}} - E_3 - E_4)$$

<u>Note</u>: $\vec{p}_4$ fixed through δ-fct.

$$\Rightarrow E_4 = \sqrt{\vec{p}_4^{\,2} + m_4^2} = \sqrt{(\vec{p}_1+\vec{p}_2-\vec{p}_3)^2 + m_4^2} \text{ is fixed here!}$$

In CM frame: $\vec{p}_1 + \vec{p}_2 = 0 \Rightarrow E_4 = \sqrt{\vec{p}_3^{\,2} + m_4^2} \equiv \sqrt{\vec{p}_*^{\,2} + m_4^2}$

$\vec{p}_*$: final-state CM momentum

$$d^3 p_3 = |\vec{p}_*|^2 \, d\Omega^* \, d|\vec{p}_*|$$

$$\Rightarrow \frac{d\sigma}{d\Omega^*} = \frac{1}{8\pi^2 \lambda^{1/2}(s, m_1^2, m_2^2)} \cdot \int \frac{|\vec{p}_*|^2 d|\vec{p}_*|}{4\sqrt{(\vec{p}_*^{\,2}+m_3^2)(\vec{p}_*^{\,2}+m_4^2)}} \cdot |F|^2 \cdot \delta\left(\sqrt{s} - \sqrt{\vec{p}_*^{\,2}+m_3^2} - \sqrt{\vec{p}_*^{\,2}+m_4^2}\right)$$

The argument of the δ-fct vanishes for physical configurations (all particles are on-shell, 4-mom. is conserved). Again using the result from 1c), this happens for $|\vec{p}_*| = \frac{\sqrt{s}}{2} \lambda^{1/2}\left(1, \frac{m_3^2}{s}, \frac{m_4^2}{s}\right) = \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{2\sqrt{s}}$

The derivative of the argument of the δ-fct w.r.t. $|\vec{p}^*|$

$$= -\frac{|\vec{p}_*|}{\sqrt{\vec{p}_*^{\,2}+m_3^2}} - \frac{|\vec{p}_*|}{\sqrt{\vec{p}_*^{\,2}+m_4^2}} \qquad \left(\text{recall: } \int \delta(f(x)) = \frac{1}{|f'(x_0)|} \quad \text{with } f(x_0)=0 \right)$$

$$= -\frac{|\vec{p}_*|(E_3+E_4)}{E_3 E_4} = \frac{|\vec{p}_*|\sqrt{s}}{E_3 E_4}$$

Hence: $$\frac{d\sigma}{d\Omega^*} = \frac{1}{8\pi^2 \lambda^{1/2}(s,m_1^2,m_2^2)} \cdot \frac{|\vec{p}_*|^2}{4\cancel{E_3E_4}} \cdot \frac{\cancel{E_3E_4}}{|\vec{p}_*|\sqrt{s}} \cdot |F|^2$$

$$= \frac{1}{32\pi^2 \lambda^{1/2}(s,m_1^2,m_2^2)} \cdot \underbrace{\frac{|\vec{p}_*|}{\sqrt{s}}}_{\frac{\lambda^{1/2}(s,m_3^2,m_4^2)}{2s}} \cdot |F|^2$$

$$= \frac{|F|^2 \cdot \lambda^{1/2}(s,m_3^2,m_4^2)}{64\pi^2 \lambda^{1/2}(s,m_1^2,m_2^2) \cdot s} \quad \checkmark\checkmark$$

2c) For cylinder symmetry around beam axis: can trivially perform ϕ^* integral in $d\Omega^* = d\phi^* \cdot d(\cos\theta^*)$

$$\Rightarrow \frac{d\sigma}{d\cos\theta^*} = \frac{|F|^2 \lambda^{1/2}(s,m_3^2,m_4^2)}{32\pi \lambda^{1/2}(s,m_1^2,m_2^2) \cdot s}$$

$$\parallel$$

$$\frac{d\sigma}{dt} \underbrace{\frac{dt}{d\cos\theta^*}}$$

from bottom of p. S3.2: $\frac{dt}{d\cos\theta^*} = 2|\vec{p}_1||\vec{p}_3|$

$$= \frac{1}{2s} \lambda^{1/2}(s,m_1^2,m_2^2)\, \lambda^{1/2}(s,m_3^2,m_4^2)$$

$$\Rightarrow \frac{d\sigma}{dt} = \frac{d\sigma}{d\cos\theta^*} \cdot \frac{2s}{\lambda^{1/2}(s,m_1^2,m_2^2)\, \lambda^{1/2}(s,m_3^2,m_4^2)}$$

$$= \frac{|F|^2}{16\pi \lambda(s,m_1^2,m_2^2)} \quad \checkmark\checkmark$$

$\uparrow$

not $\lambda^{\frac{1}{2}}$!

3.1) $\gamma_\mu \gamma^\mu = \underbrace{g_{\mu\nu}}_{\text{symm.}} \gamma^\nu \gamma^\mu = \frac{1}{2} g_{\mu\nu} \underbrace{(\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu)}_{\text{symmetrized!}}$

$= \frac{1}{2} g_{\mu\nu} \{\gamma^\nu, \gamma^\mu\} = g_{\mu\nu} g^{\mu\nu} = 4 \checkmark \; (\cdot \mathbb{1}_{4\times4}$: suppressed$)$

3.2) $\gamma_\mu a\!\!\!/ \gamma^\mu = a_\nu \gamma_\mu \underbrace{\gamma^\nu \gamma^\mu}_{\text{anti-commute}} = a_\nu \gamma_\mu (-\gamma^\mu \gamma^\nu + \{\gamma^\mu, \gamma^\nu\})$

$= -a_\nu \underbrace{\gamma_\mu \gamma^\mu}_{4} \gamma^\nu + a_\nu \gamma_\mu \cdot 2 g^{\mu\nu}$

$= -4 a_\nu \gamma^\nu + 2 a_\nu \gamma^\nu = -2 a\!\!\!/ \quad \checkmark\checkmark$

3.3) $\gamma_\mu a\!\!\!/ b\!\!\!/ \gamma^\mu = a_\alpha b_\beta \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu$

$= a_\alpha b_\beta \gamma_\mu \gamma^\alpha (-\gamma^\mu \gamma^\beta + 2 g^{\beta\mu})$

$= -a_\alpha b_\beta \underbrace{\gamma_\mu \gamma^\alpha \gamma^\mu}_{-2\gamma^\alpha \text{ (s. 3.2)}} \gamma^\beta + 2 a_\alpha b_\beta \gamma^\beta \gamma^\alpha$

$= 2 a_\alpha b_\beta \underbrace{(\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha)}_{2 g^{\alpha\beta}} = 4 a \cdot b \quad \checkmark\checkmark$

3.4) $\gamma_\mu a\!\!\!/ b\!\!\!/ c\!\!\!/ \gamma^\mu = a_\alpha b_\beta c_\gamma \gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\mu$

$= a_\alpha b_\beta c_\gamma \gamma_\mu \gamma^\alpha \gamma^\beta (-\gamma^\mu \gamma^\gamma + 2 g^{\gamma\mu})$

$= a_\alpha b_\beta c_\gamma \cdot (-\underbrace{\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu}_{4 g^{\alpha\beta} \text{ s. (3.3)}} \gamma^\gamma + 2 \gamma^\gamma \gamma^\alpha \gamma^\beta)$

$= a_\alpha b_\beta c_\gamma (\cancel{-4 g^{\alpha\beta} \gamma^\gamma} - 2 \gamma^\gamma \gamma^\beta \gamma^\alpha \cancel{+ 4 \gamma^\gamma g^{\alpha\beta}}) = -2 c\!\!\!/ b\!\!\!/ a\!\!\!/ \quad \checkmark\checkmark$

S3.8

3.5) $\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}) = \mathrm{tr}(\underbrace{\gamma_5\gamma_5}_{1\!\!1}\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n})$

anti-com: $\{\gamma_5, \gamma^\mu\} = 0$

$= \mathrm{tr}(\gamma_5\gamma^{\mu_1}\gamma_5\gamma^{\mu_2}\dots\gamma^{\mu_n})$ Keep anti-commuting the γ_5 until the end

$= (-1)^n\,\mathrm{tr}(\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}\gamma_5)$ move back to front

$= (-1)^n\,\mathrm{tr}(\gamma_5\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}) = (-1)^n\,\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n})$

Hence either $(-1)^n = 1$ (i.e. n is even), or $\mathrm{tr}(\gamma^{\mu_1}\dots\gamma^{\mu_n}) = 0$ (for n odd)

Used: $\mathrm{tr}(AB) = \sum_i (AB)_{ii} = \sum_{i,j} A_{ij}B_{ji} = \sum_{i,j} B_{ji}A_{ij} = \sum_j (BA)_{jj} = \mathrm{tr}(BA)$

$\mathbb{C}$-numbers commute!

3.6) $\mathrm{tr}(\not a \not b) = a_\alpha b_\beta\,\mathrm{tr}(\gamma^\alpha\gamma^\beta) = \frac{1}{2} a_\alpha b_\beta\,\mathrm{tr}(\underbrace{\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha}_{2g^{\alpha\beta}\cdot 1\!\!1_{4\times4}})$

$= \frac{1}{2} a_\alpha b_\beta\, 2 g^{\alpha\beta}\cdot\underbrace{\mathrm{tr}\,1\!\!1_{4\times4}}_{4} = 4\, a\cdot b$ ✓✓

3.7) $\mathrm{tr}(\not a \not b \not c \not d) = a_\alpha b_\beta c_\gamma d_\delta\,\mathrm{tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta)$ $\quad 4g^{\alpha\beta} = (3.6)$

$= a_\alpha b_\beta c_\gamma d_\delta\cdot\{-\mathrm{tr}(\gamma^\alpha\gamma^\beta\gamma^\delta\gamma^\gamma) + 2g^{\gamma\delta}\,\mathrm{tr}(\gamma^\alpha\gamma^\beta)\}$

$= a_\alpha b_\beta c_\gamma d_\delta\cdot\{\mathrm{tr}(\gamma^\alpha\gamma^\delta\gamma^\beta\gamma^\gamma) - 2g^{\beta\delta}\underbrace{\mathrm{tr}(\gamma^\alpha\gamma^\gamma)}_{4g^{\alpha\gamma}} + 8g^{\gamma\delta}g^{\alpha\beta}\}$

move back

$= a_\alpha b_\beta c_\gamma d_\delta\cdot\{-\mathrm{tr}(\gamma^\delta\gamma^\alpha\gamma^\beta\gamma^\gamma) + 2g^{\alpha\delta}\underbrace{\mathrm{tr}(\gamma^\beta\gamma^\gamma)}_{4g^{\beta\gamma}} - 8g^{\beta\delta}g^{\alpha\gamma}$
$+ 8g^{\gamma\delta}g^{\alpha\beta}\}$

S3.9

In the first term, we can move γ^δ all the way to the end again under the trace, using $tr(AB) = tr(BA)$ with $A = \gamma^\delta$, $B = \gamma^\alpha\gamma^\beta\gamma^\delta$

$$\Rightarrow tr(\not{a}\not{b}\not{c}\not{d}) = -tr(\not{a}\not{b}\not{c}\not{d}) + 8(a\cdot d)(b\cdot c) - 8(b\cdot d)(a\cdot d) + 8(c\cdot d)(a\cdot b)$$

$$\Rightarrow 2tr(\not{a}\not{b}\not{c}\not{d}) = 8(a\cdot d)(b\cdot c) - 8(a\cdot c)(b\cdot d) + 8(a\cdot b)(c\cdot d)$$

$\Rightarrow$ q.e.d.