4th Sheet

1) In chiral basis, had

$$u = \begin{pmatrix} u_- \\ u_+ \end{pmatrix} = \begin{pmatrix} \omega_{-d}\, \kappa_d \\ \omega_d\, \kappa_d \end{pmatrix} \quad ; \quad v = \begin{pmatrix} v_- \\ v_+ \end{pmatrix} = \begin{pmatrix} d\, \omega_d\, \kappa_{-d} \\ -d\, \omega_{-d}\, \kappa_{-d} \end{pmatrix}$$

$d = \pm 1$ labels the spin states.

$$\kappa_+ = \frac{1}{N} \begin{pmatrix} |\vec{p}| + p_z \\ p_x + i p_y \end{pmatrix} \qquad \left(N = \sqrt{2|\vec{p}|(|\vec{p}| + p_z)} \right)$$

$$\kappa_- = \frac{1}{N} \begin{pmatrix} -p_x + i p_y \\ |\vec{p}| + p_z \end{pmatrix} \qquad \omega_\pm = \sqrt{E \pm |\vec{p}|} \in \mathbb{R}$$

1a) $-i\gamma^2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}$

$$\Rightarrow -i\gamma^2 u^* = \begin{pmatrix} -i\sigma_2\, \omega_d\, \kappa_d^* \\ i\sigma_2\, \omega_{-d}\, \kappa_d^* \end{pmatrix} \quad ; \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} \Rightarrow i\sigma_2 \kappa_+^* &= \frac{1}{N} \begin{pmatrix} p_x - i p_y \\ -|\vec{p}| - p_z \end{pmatrix} = -\kappa_- \\ i\sigma_2 \kappa_-^* &= \frac{1}{N} \begin{pmatrix} |\vec{p}| + p_z \\ p_x + i p_y \end{pmatrix} = \kappa_+ \end{aligned}\right\} \Rightarrow \underline{i\sigma_2 \kappa_d^* = -d\, \kappa_{-d}}$$

$$\Rightarrow -i\gamma^2 u^* = \begin{pmatrix} \omega_d\, d\, \kappa_{-d} \\ -\omega_{-d}\, d\, \kappa_{-d} \end{pmatrix} = v \quad \checkmark\checkmark$$

Note that $\mathcal{C}^2 = 1$: $\mathcal{C}(\mathcal{C}\psi) = \mathcal{C}(-i\gamma_2\psi^*) = \overbrace{-\gamma_2\gamma_2}^{1}\overbrace{(\psi^*)^*}^{\psi} = \psi$

$$\Rightarrow \mathcal{C} v = \mathcal{C}^2 u = u \quad \checkmark$$

[54.2]

so that $p^0 > 0$!

Fourier decomposition of ψ:

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left[u^s(\vec{p}) e^{-ip\cdot x} + v^s(\vec{p}) e^{ip\cdot x} \right]$$

Under P: $x^0 = t \to x^0$, $\vec{x} \to -\vec{x}$ $\Rightarrow$ $p\cdot x = E_{\vec{p}}t - \vec{p}\cdot\vec{x} \xrightarrow{P} E_p t + \vec{p}\cdot\vec{x} \equiv \tilde{p}\cdot x$,

with $\tilde{p} = (E_p, -\vec{p})$

$\Rightarrow$ can change integration variable:

$$\psi \xleftrightarrow{P} \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{\vec{p}}}}} \gamma^0 \sum_s \left[u^s(\tilde{\vec{p}}) e^{-i\tilde{p}\cdot x} + v^s(\tilde{\vec{p}}) e^{i\tilde{p}\cdot x} \right]$$

$$E_{\tilde{\vec{p}}} = E_{\vec{p}}!, \qquad \tilde{\vec{p}} = -\vec{p}$$

A P trafo on the spinors u, v therefore implies $\vec{p} \to -\vec{p}$.
In addition, we have to multiply with γ^0:

$$\gamma^0 u(\tilde{\vec{p}}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_{-\lambda}\chi_\lambda(\tilde{\vec{p}}) \\ w_\lambda \chi_\lambda(\tilde{\vec{p}}) \end{pmatrix} = \begin{pmatrix} w_\lambda \chi_\lambda(\tilde{\vec{p}}) \\ w_{-\lambda}\chi_\lambda(\tilde{\vec{p}}) \end{pmatrix}$$

chiral basis

$$\gamma^0 v(\tilde{\vec{p}}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda w_\lambda \chi_{-\lambda}(\tilde{\vec{p}}) \\ -\lambda w_{-\lambda}\chi_{-\lambda}(\tilde{\vec{p}}) \end{pmatrix} = (-1)\cdot \begin{pmatrix} \lambda w_{-\lambda}\chi_{-\lambda}(\tilde{\vec{p}}) \\ -\lambda w_\lambda \chi_{-\lambda}(\tilde{\vec{p}}) \end{pmatrix}$$

extra minus sign!

Note: λ is projection of the spin on $\vec{p}$. In order to keep the spin fixed when $\vec{p} \to -\vec{p}$: need $\lambda \to -\lambda$, i.e. swap $w_\lambda \leftrightarrow w_{-\lambda}$!
Alternative derivation: Peskin & Schroeder, p. 65.

A T transformation changes $\vec{p} \to -\vec{p}$, but leaves $\vec{x}$ invariant. Hence angular momentum, and spin, also should flip sign (but helicity does not!)

b) First, rewrite C trafo:

$$\psi \xrightarrow{C} -i\gamma^2\psi^* = -i\gamma^2\psi^{+T} = -i\gamma^2(\psi^+\underbrace{\gamma^0\gamma^0}_{1})^T = -i\underset{\gamma^2}{\gamma}\gamma^{0T}\overbrace{\bar\psi^T}^{\bar\psi}$$

$$= -i(\bar\psi\gamma^0\gamma^{2T})^T = -i(\bar\psi\gamma^0\gamma^2)^T$$

$$\bar\psi = \psi^+\gamma^0 \to i\psi^T\underset{-\gamma^2}{\gamma^{2+}}\gamma^0 = -i\psi^T\gamma^2\gamma^0 = -i(\underset{\gamma^0}{\gamma^{0T}}\underset{\gamma^2}{\gamma^{2T}}\psi)^T = -i(\gamma^0\gamma^2\psi)^T$$

Used: $\gamma^0 = \gamma^{0T} = \gamma^{0+}$; $\gamma^2 = \gamma^{2T} = -\gamma^{2+}$ (from explicit rep.)

Thus: $\bar\psi\psi \xrightarrow{C} -(\gamma^0\gamma^2\psi)^T(\bar\psi\gamma^0\gamma^2)^T \overset{\psi,\bar\psi\text{ anti-commute}}{=} +(\bar\psi\gamma^0\gamma^2\gamma^0\gamma^2\psi)^T$

$$= -(\bar\psi\underbrace{\gamma^0\gamma^0}_{1}\underbrace{\gamma^2\gamma^2}_{-1}\psi)^T = \underbrace{(\bar\psi\psi)^T}_{\text{no free Dirac index!}} = \bar\psi\psi$$

$$P: \psi(t,\vec x) \xrightarrow{P} \gamma^0\psi(t,-\vec x) \Rightarrow \bar\psi(t,\vec x) = \psi^+(t,\vec x)\gamma^0 \xrightarrow{P} \psi^+(t,-\vec x)\overbrace{\gamma^{0+}}^{\gamma^0}\gamma^0$$

$$= \bar\psi(t,-\vec x)\gamma^0$$

$$\Rightarrow \bar\psi(t,\vec x)\psi(t,\vec x) \xrightarrow{P} \bar\psi(t,-\vec x)\gamma^0\gamma^0\psi(t,-\vec x) = \bar\psi(t,-\vec x)\psi(t,-\vec x)$$

The change $\vec x \to -\vec x$ just redefines the coordinate: not important

$\Rightarrow \bar\psi\psi$ is also P invariant.

$$T: \psi(t,\vec x) \xrightarrow{T} -\gamma^1\gamma^3\psi(-t,\vec x)$$

$$\Rightarrow \bar\psi(t,\vec x) = \psi^+(t,\vec x)\gamma^0 \xrightarrow{T} -\psi^+(-t,\vec x)\underset{-\gamma^3}{\gamma^{3+}}\underset{-\gamma^1}{\gamma^{1+}}\gamma^0 = -\psi^+(-t,\vec x)\gamma^3\gamma^1\gamma^0$$

$$= -\psi^+(-t,\vec x)\gamma^0\gamma^3\gamma^1 = -\bar\psi(-t,\vec x)\gamma^3\gamma^1$$

$$\Rightarrow \bar\psi(t,\vec x)\psi(t,\vec x) \to \bar\psi(-t,\vec x)\gamma^3\underbrace{\gamma^1\gamma^1}_{-1}\gamma^3\psi(-t,\vec x) = -\bar\psi(-t,\vec x)\overbrace{\gamma^3\gamma^3}^{-1}\psi(-t,\vec x)$$

$$= \bar\psi(-t,\vec x)\psi(-t,\vec x)$$

The change $t \to -t$ is again immaterial in the action.

Before tackling the kinetic energy term, consider $\bar\psi\gamma^\mu\psi$ w/o derivative!

$$\bar\psi(t,\vec{x})\gamma^\mu\psi(t,\vec{x}) \xrightarrow{C} -(\gamma^0\gamma^2\psi)^T\gamma^\mu(\bar\psi\gamma^0\gamma^2)^T$$

$$= -\psi^T \underbrace{\gamma^{2T}}_{=\gamma^2}\underbrace{\gamma^{0T}}_{=\gamma^0}\gamma^\mu \underbrace{\gamma^{2T}}_{=\gamma^2}\underbrace{\gamma^{0T}}_{=\gamma^0}\bar\psi^T$$

$$= \psi^T\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^2\bar\psi^T$$

Need to compute $\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^2$:

$\mu=0$: $= \gamma^2\gamma^0\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\gamma^2 = \gamma^2\gamma^0\gamma^2 = -\overbrace{\gamma^2\gamma^2}^{-\mathbb{1}}\gamma^0 = \gamma^0 = \gamma^{0T}$

$\mu=2$: $= \gamma^2\gamma^0\gamma^2\gamma^0\gamma^2 = -\gamma^2\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\underbrace{\gamma^2\gamma^2}_{-\mathbb{1}} = \gamma^2 = \gamma^{2T}$

$\mu=1,3$: $\gamma^2\gamma^0\gamma^{1,3}\gamma^0\gamma^2 = \gamma^2\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\gamma^2\gamma^{1,3} = \gamma^2\gamma^2\gamma^{1,3} = -\gamma^{1,3} = \gamma^{1,3T}$

due anti-sym!

Hence: $\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^2 = \gamma^{\mu T}\ \forall\mu$!

$\psi, \bar\psi$ anti-com.

So: $\bar\psi(t,\vec{x})\gamma^\mu\psi(t,\vec{x}) \xrightarrow{C} \psi^T\gamma^{\mu T}\bar\psi^T = -\underbrace{(\bar\psi\gamma^\mu\psi)^T}_{\text{no free Dirac index}} = -\bar\psi\gamma^\mu\psi$;

This vector current changes sign under C!

$$\bar\psi(t,\vec{x})\gamma^\mu\psi(t,\vec{x}) \xrightarrow{P} \bar\psi(t,-\vec{x})\gamma^0\gamma^\mu\gamma^0\psi(t,-\vec{x})$$

$$= \begin{cases} \bar\psi(t,-\vec{x})\gamma^0\psi(t,-\vec{x}) & : \mu=0,\ \underline{\text{no}}\ \text{change of sign} \\ -\bar\psi(t,-\vec{x})\gamma^i\psi(t,-\vec{x}) & : \mu=i\in\{1,2,3\};\ \text{change of sign} \end{cases}$$

[S4.5]

This is as expected for a 4-vector: the $\mu=0$ component (time, energy) does not change when $\vec{x}\to-\vec{x}$, the spatial components do!

$$\bar\Psi(t,\vec{x})\gamma^\mu\Psi(t,\vec{x}) \xrightarrow[T]{} \bar\Psi(-t,\vec{x})\gamma^3\gamma^1\gamma^{\mu *}\gamma^1\gamma^3\Psi(-t,\vec{x})$$

T also includes complex conjugation!

Need $\gamma^3\gamma^1\gamma^{\mu *}\gamma^1\gamma^3$

$\mu=0$: $\gamma^{0*}=\gamma^0$: $\gamma^3\gamma^1\gamma^0\gamma^1\gamma^3 = \gamma^3\underbrace{\gamma^1\gamma^1}_{-1}\gamma^3\gamma^0 = \gamma^0$

$\mu=1$: $\gamma^{1*}=\gamma^1$: $\gamma^3\gamma^1\underbrace{\gamma^1\gamma^1}_{-1}\gamma^3 = -\gamma^3\gamma^1\gamma^3 = \gamma^3\gamma^3\gamma^1 = -\gamma^1$

$\mu=2$: $\gamma^{2*}=-\gamma^2$: $-\gamma^3\gamma^1\gamma^2\gamma^1\gamma^3 = -\gamma^3\gamma^1\gamma^1\gamma^3\gamma^2 = -\gamma^2$

$\mu=3$: $\gamma^{3*}=\gamma^3$: $\gamma^3\gamma^1\gamma^3\gamma^1\gamma^3 = -\gamma^3\underbrace{\gamma^1\gamma^1}_{-1}\underbrace{\gamma^3\gamma^3}_{-1} = -\gamma^3$

Hence: $\gamma^3\gamma^1\gamma^{\mu *}\gamma^1\gamma^3 = \begin{cases}\gamma^0, & \mu=0\\ -\gamma^i, & \mu=i\end{cases}$

$$\Rightarrow \bar\Psi(t,\vec{x})\gamma^\mu\Psi(t,\vec{x}) \to \begin{cases}\bar\Psi(-t,\vec{x})\gamma^0\Psi(-t,\vec{x}), & \mu=0\\ -\bar\Psi(-t,\vec{x})\gamma^i\Psi(-t,\vec{x}), & \mu=i\end{cases}$$

Again, only the spatial components change sign: as for a 4-current, eg. electric current!

Now let's include the derivative: From p. S4.3:

$$i\partial_\mu\Psi \xrightarrow[C]{} \partial_\mu(\bar\Psi\gamma^0\gamma^2)^T = (\partial_\mu\bar\Psi\gamma^0\gamma^2)^T$$

$$\Rightarrow \bar{\Psi} i\gamma^\mu \partial_\mu \Psi \xrightarrow[C]{} i\Psi^T \gamma^2 \gamma^0 \gamma^\mu \gamma^0 \gamma^2 \partial_\mu \bar{\Psi}^T$$

$$= i(\Psi^T \gamma^{\mu T} \partial_\mu \bar{\Psi}^T) \underset{\substack{\uparrow \\ \Psi, \partial_\mu \bar{\Psi} \\ \text{anti-comm.}}}{=} -i(\partial_\mu \bar{\Psi}) \gamma^\mu \Psi$$

$$= i\bar{\Psi}\partial_\mu \gamma^\mu \Psi + \underbrace{(\text{total derivative})}_{\text{ignore}}$$

$$\bar{\Psi} i\gamma^\mu \partial_\mu \Psi \xrightarrow[P]{} \begin{cases} \bar{\Psi}(t,-\vec{x})\, i\gamma^0 \partial_0 \Psi(t,-\vec{x}), & \mu = 0 \\ -\bar{\Psi}(t,-\vec{x})(-i\gamma^i \partial_i)\Psi(t,-\vec{x}) \\ \quad = \bar{\Psi}(t,-\vec{x})\, i\gamma^i \partial_i \Psi(t,-\vec{x}) \end{cases}$$

$$= \bar{\Psi} i\gamma^\mu \partial_\mu \Psi$$

The extra minus sign in the P trafo of the spatial comp. of $\bar{\Psi}\gamma^\mu\Psi$ (see p. S4.4, bottom) is compensated by the flip of sign of the spatial derivative under P $\Rightarrow$ term is invariant (up to irrelevant ~~re~~-definition of $\vec{x}$, again)

Under a T trafo: $i \xrightarrow[T]{} -i$; $\partial_0 \xrightarrow[T]{} -\partial_0$; $\partial_k \xrightarrow[T]{} +\partial_k$

$$\Rightarrow i\partial_\mu \rightarrow \begin{cases} +i\partial_0, & \mu = 0 \\ -i\partial_k, & \mu = k \in \{1,2,3\} \end{cases}$$

For $\mu = k$, i.e. spatial index, this again compensates the sign flip of $\bar{\Psi}\gamma^\mu\Psi$, see p. S4.5.

2c): Already done!

2) Is most easily shown using explicit solutions of the free Dirac eq. E.g. using the Dirac rep:

$$u(p,s) = \sqrt{E+m}\begin{pmatrix} \phi_s \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi_s \end{pmatrix} \Rightarrow \bar{u}(p,s) = \sqrt{E+m}\left(\phi_s^\dagger, \overset{\text{from } \gamma^0: \bar{u} = u^\dagger\gamma^0}{-\phi_s^\dagger}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)$$

$\phi_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\phi_2 = \begin{pmatrix}0\\1\end{pmatrix}$ are the two spin states

$$(\vec{p}\cdot\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} p_z \\ p_x + ip_y\end{pmatrix}$$

$$(\vec{p}\cdot\vec{\sigma})\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} p_x - ip_y \\ -p_z\end{pmatrix}$$

$$(1,0)(\vec{p}\cdot\vec{\sigma}) = (p_z, p_x - ip_y); \quad \overset{\phi_2^\dagger}{(0,1)}(\vec{p}\cdot\vec{\sigma}) = (p_x + ip_y, -p_z)$$

$$\Rightarrow \sum_s u(p,s)\bar{u}(p,s) = (E+m)\cdot\left\{ \overset{s=1}{\begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x+ip_y)/(E+m)\end{pmatrix}\cdot\left(1, 0, \frac{-p_z}{E+m}, \frac{-p_x+ip_y}{E+m}\right)} \right.$$

$$\left. + \underbrace{\begin{pmatrix} 0 \\ 1 \\ (p_x-ip_y)/(E+m) \\ -p_z/(E+m)\end{pmatrix}\cdot\left(0, 1, \frac{-p_x-ip_y}{E+m}, \frac{p_z}{E+m}\right)}_{4\times 4 \text{ matrices!}} \right\}$$

S6.8

$$\Rightarrow \sum_s u(p,s)\,\bar{u}(p,s) = (E+m)\cdot\begin{pmatrix} 1, & 0, & \frac{-p_z}{E+m}, & \frac{-p_x+ip_y}{E+m} \\ 0, & 1, & \frac{-p_x-ip_y}{E+m}, & \frac{p_z}{E+m} \\ \frac{p_z}{E+m}, & \frac{p_x-ip_y}{E+m}, & \frac{-\vec{p}^2}{(E+m)^4} & 0 \\ \frac{p_x+ip_y}{E+m}, & \frac{-p_z}{E+m}, & 0, & -\frac{\vec{p}^2}{(E+m)^2} \end{pmatrix}$$

$$= \begin{pmatrix} E+m, & 0, & -p_z, & -p_x+ip_y \\ 0 & , E+m, & -p_x-ip_y, & \cancel{p} \; p_z \\ p_z & p_x-ip_y, & -\vec{p}^2/E+m, & 0 \\ p_x+ip_y & -p_z & , 0 & , -\frac{\vec{p}^2}{E+m} \end{pmatrix}$$

$$-\frac{\vec{p}^2}{E+m} = -\frac{E^2-m^2}{E+m} = -(E-m) = -E+m$$

$$\Rightarrow \sum_s u(p,s)\,\bar{u}(p,s) = \begin{pmatrix} (E+m)\mathbb{1}_{2\times2} & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & (-E+m)\mathbb{1}_{2\times2} \end{pmatrix} = (\not{p}+m) \;\checkmark\checkmark$$

$$\not{p} = p_\mu \gamma^\mu = E\gamma^0 - \vec{p}\cdot\vec{\sigma}$$

$$= \begin{pmatrix} E & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E \end{pmatrix}$$

Similarly:

$$v = \sqrt{E+m}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_s \\ \chi_s \end{pmatrix} \Rightarrow \bar{v} = \sqrt{E+m}\left(\chi_s^{+}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}, -\chi_s^{+}\right)$$

$$\chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \phi_2, \quad \chi_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\phi_1$$

This gives:

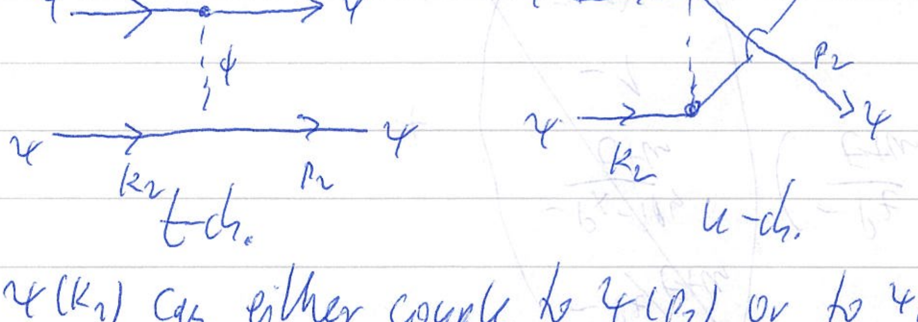
S4.9

$$\Rightarrow \sum_s v(p,s)\,\bar v(p,s) = (E+m)\left\{\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\ \frac{-p_z}{E+m}\\ 0\\ 1\end{pmatrix}\cdot\left(\frac{p_x+ip_y}{E+m}, \frac{-p_z}{E+m}, 0, -1\right) + \begin{pmatrix}\frac{-p_z}{E+m}\\ \frac{-p_x-ip_y}{E+m}\\ -1\\ 0\end{pmatrix}\left(\frac{-p_z}{E+m}, \frac{-p_x+ip_y}{E+m}, 1, 0\right)\right\}$$

$$= \begin{pmatrix} \vec p^{\,2}/(E+m), & 0, & -p_z, & -p_x+ip_y \\ 0, & \vec p^{\,2}/(E+m), & -p_x-ip_y, & p_z \\ p_z, & p_x-ip_y, & -E-m, & 0 \\ p_x+ip_y, & -p_z, & 0, & -E-m \end{pmatrix} \qquad \frac{\vec p^{\,2}}{E+m} = \frac{E^2-m^2}{E+m} = E-m$$

$\Rightarrow \sum_s v\bar v$ differs from $\sum_s u\bar u$ only by $m \to -m$ ✓✓

3)
a)



$\psi(k_1)$ can either couple to $\psi(p_1)$ or to $\psi(p_2)$. Since the two diagrams are related by swapping fermion line: rel. relative -1!

b) $\Rightarrow F = \lambda^2 \cdot \left\{ \frac{1}{t-m_\phi^2}\,\bar u(p_1)\,u(k_1)\,\bar u(p_2)\,u(k_2) - \frac{1}{u-m_\phi^2}\,\bar u(p_2)\,u(k_1)\,\bar u(p_1)\,u(k_2)\right\}$: is evidently

c) anti-sym. under $p_1 \leftrightarrow p_2$, or $k_1 \leftrightarrow k_2$!

3 d) $|\bar{F}|^2 = \frac{\lambda^4}{4}\cdot\Big\{ \frac{1}{(t-m_\varphi^2)^2}\cdot tr[(\not k_1 + m_\psi)(\not p_1 + m_\psi)]\cdot tr[(\not k_2+m_\psi)(\not p_2+m_\psi)]$

$+\frac{1}{(u-m_\varphi^2)^2}\cdot tr[(\not k_1+m_\psi)(\not p_2+m_\psi)]\cdot tr[(\not k_2+m_\psi)(\not p_1+m_\psi)]$

$-\frac{2}{(t-m_\varphi^2)(u-m_\varphi^2)}\, Re\Big[\underbrace{tr[(\not k_1+m_\psi)(\not p_1+m_\psi)(\not k_2+m_\psi)(\not p_2+m_\psi)]}\Big]\Big\}$

$= 4[\, k_1\cdot p_2\, k_2\cdot p_1 + k_1\cdot p_1\, p_2\cdot k_2 - k_1\cdot k_2\, p_1\cdot p_2$
$+ m_\psi^2\cdot(k_1\cdot p_2 + k_1\cdot k_2 + k_1\cdot p_1 + p_2\cdot k_2 + p_2\cdot p_1 + p_1\cdot k_2) + m_\psi^4]$

Since masses in the initial and final state are the same, the kinematics is simple: $\sqrt{1-4m_\psi^2/s}$

$k_1 = \frac{\sqrt{s}}{2}(1,0,0,\beta)\ ;\ k_2 = \frac{\sqrt{s}}{2}(1,0,0,-\beta);$

$p_1 = \frac{\sqrt{s}}{2}(1,\ldots,\beta\cos\theta^*)\ ;\ p_2 = \frac{\sqrt{s}}{2}(1,\ldots,-\beta\cos\theta^*)$

$\Rightarrow k_1\cdot k_2 = p_1\cdot p_2 = \frac{s}{2} - m_\psi^2 = \frac{s}{4}(1+\beta^2)$

$k_1\cdot p_1 = k_2\cdot p_2 = \frac{s}{4}(1-\beta^2\cos\theta^*)\ ;\ k_2\cdot p_1 = k_1\cdot p_2 = \frac{s}{4}(1+\beta^2\cos\theta^*)$

$|\bar{F}|^2 = \frac{\lambda^4}{4}\cdot\Big\{\frac{16}{(t-m_\varphi^2)^2}\cdot[k_1\cdot p_1 + m_\psi^2]\cdot[k_2\cdot p_2 + m_\psi^2]$

$+\frac{16}{(u-m_\varphi^2)^2}\cdot[k_1\cdot p_2 + m_\psi^2][k_2\cdot p_1 + m_\psi^2]$

$-\frac{8}{(t-m_\varphi^2)(u-m_\varphi^2)}\cdot[\,k_1\cdot p_2\, k_2\cdot p_1 + k_1\cdot p_1\, p_2\cdot k_2 - k_1\cdot k_2\, p_1\cdot p_2$
$+ m_\psi^2(k_1\cdot p_2 + k_1\cdot k_2 + k_1\cdot p_1 + p_2\cdot k_2 + p_2\cdot p_1 + p_1\cdot k_2) + m_\psi^4]\Big\}$

S4.11

$$|F|^2 = \lambda^4 \cdot \Big\{ \frac{4}{(t-m_\phi^2)^2} \cdot \Big[\frac{s^2}{16}(1-\beta^2\cos\theta^*)^2 + \frac{s m_\phi^2}{2}(1-\beta^2\cos\theta^*) + m_\phi^4 \Big]$$

$$+ \frac{4}{(u-m_\phi^2)^2} \cdot \Big[\frac{s^2}{16}(1+\beta^2\cos\theta^*)^2 + \frac{s m_\phi^2}{2}(1+\beta^2\cos\theta^*) + m_\phi^4 \Big]$$

$$- \frac{2}{(t-m_\phi^2)(u-m_\phi^2)} \cdot \Big[\frac{s^2}{16}(1+\beta^2\cos\theta^*)^2 + \frac{s^2}{16}(1-\beta^2\cos\theta^*)^2 - \frac{s^2}{16}(1+\beta^2)^2$$

$$+ m_\phi^4 + m_\phi^2 \cdot \underbrace{\Big(k_1 \cdot \underbrace{(p_1+p_2+k_2)}_{2k_2+k_1} + k_2 \underbrace{(p_1+p_2)}_{k_1+k_2} + p_1 \cdot p_2 \Big)}_{} \Big] \Big\}$$

$$3k_1 \cdot k_2 + 2m_\phi^2 + p_1 \cdot p_2$$

$$= \frac{3s}{2} - 3m_\phi^2 + 2m_\phi^2 + \frac{s}{2} - m_\phi^2 = 2s - 2m_\phi^2$$

$$t = (p_1-k_1)^2 = 2m_\phi^2 - 2p_1 \cdot k_1 = 2m_\phi^2 - \frac{s}{2}(1-\beta^2\cos\theta^*)$$

$$u = (p_2-k_1)^2 = 2m_\phi^2 - \frac{s}{2}(1+\beta^2\cos\theta^*)$$

$$|F|^2 = \lambda^4 \cdot \Big\{ \frac{\cancel{4}\,1}{(t-m_\phi^2)^2} \cdot \Big[\frac{s^2}{\cancel{4}}(1-\beta^2\cos\theta^*)^2 + 2s\, m_\phi^2 (1-\beta\cos\theta^*) + 4m_\phi^4 \Big]$$

$$+ \frac{1}{(u-m_\phi^2)^2} \cdot \Big[\frac{s^2}{4}(1+\beta^2\cos\theta^*)^2 + 2s\, m_\phi^2(1+\beta\cos\theta^*) + 4m_\phi^4 \Big]$$

$$- \frac{1}{(t-m_\phi^2)(u-m_\phi^2)} \cdot \Big[\frac{s^2}{4}(1+\beta^4\cos^2\theta^*) - \frac{s^2}{8}(1+\beta^2)^2 + 2s\, m_\phi^2 - m_\phi^4 \Big] \Big\}$$

Note: There is no ϕ^*-dependence $\Rightarrow \int d\phi^* = 2\pi$ is trivial

$$\Rightarrow \frac{d\sigma}{d\cos\theta^*} = \frac{|F|^2}{32\pi s}$$

The full θ^*-dependence is rather complicated. Let's consider the high-$\sqrt{s}$ limit, $s \gg m_\varphi^2, m_\chi^2$, and (somewhat naively) set all masses to zero. In that limit: ($\beta \to 1$!)

$$t = -\frac{s}{2}(1-\cos\theta^*), \quad u = -\frac{s}{2}(1+\cos\theta^*) \Rightarrow u \cdot t = \frac{s^2}{4}(1-\cos^2\theta^*)$$

$$\frac{d\sigma}{d\cos\theta^*} = \frac{\lambda^4}{32\pi s} \cdot \Big\{ \frac{4}{s^2(1-\cos\theta^*)^2} \cdot \frac{s^2}{4} \cdot (1-\cos\theta^*)^2$$

$$+ \frac{4}{s^2(1+\cos\theta^*)^2} \cdot \frac{s^2}{4}(1+\cos\theta^*)^2$$

$$- \frac{4}{s^2(1-\cos^2\theta^*)} \cdot \Big[\underbrace{\frac{s^2}{4}(1+\cos^2\theta^*) - \frac{s^2}{2}}_{= \frac{s^2}{4}(-1+\cos^2\theta^*)} \Big] \Big\}$$

$$= \frac{\lambda^4}{32\pi s} \cdot 3 \Rightarrow \sigma_{tot} \xrightarrow[s \gg m_\varphi^2, m_\chi^2]{} \frac{3\lambda^4}{16\pi s}$$

Note:

*) The poles in t, u for $\cos\theta^* \to \pm 1$ get cancelled by the numerators: this is characteristic for scalar exchange, and is not true for (e.g.) photon exch. in the t- or u-channel ($\to$ later)

*) The cross section drops $\sim \frac{1}{s}$ at high s: this is always true for $2 \to 2$ scattering at fixed angle.