Crash course on "applied QFT" (September 2023)
Homework No. 1 (Sep. 25, 2023)

# 1 Lorentz transformation

A Lorentz transformation in $x^1$ direction is given by

$$t' = \gamma(t - vx^1),$$

$$x'^1 = \gamma(-vt + x^1),$$
$$x'^2 = x^2,$$
$$x'^3 = x^3$$

where $\gamma = (1 - v^2)^{-1/2}$ and $c = 1$.

1. Write down the inverse of this transformation, i.e. express $(t, x^1)$ in terms of $(t', x'^1)$.

2. Show that $\partial\phi/\partial x^\mu$ is a *covariant* four–vector, i.e. that $x^\mu\partial\phi/\partial x^\mu$ is Lorentz invariant (summation convention!). Here $\phi$ is a scalar function of $x, y, z$ and $t$. *Hint:* Use the 'chain rule', $\partial\phi/\partial x^{\mu\prime} = (\partial\phi/\partial x^\nu)(\partial x^\nu/\partial x^{\mu\prime})$.

# 2 Gamma matrices

In the Dirac representation, the Dirac matrices can be written as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where the $\sigma^i (i = 1, 2, 3)$ are the Pauli matrices, 1 denotes the $2 \times 2$ unit matrix, and 0 is the $2 \times 2$ matrix of zeros. Show that this ansatz satisfies the definining property of the Dirac matricies, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

# 3 Dirac equation

The Dirac equation can be written as

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0\,,$$

and its plane wave solutions are

$$\psi = \omega e^{-ip.x}\,.$$

Here $p^\mu = (p^0, \mathbf{p})$ is the momentum four–vector, and $\omega$ is the four–component Dirac spinor which is decomposed into two–component spinors $\phi$ and $\chi$, $\omega = N \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ , where $N$ is a normalization constant.

1. Define the conjugate spinor

   $$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$$

   where $\psi^\dagger$ is the hermitean conjugate of $\psi$, and use the covariant form of the Dirac equation to derive the adjoint Dirac equation

   $$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0\,.$$

2. Use the original and adjoint Dirac equations to show that the Dirac probability current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved.

3. With the given normalization constant $N = \sqrt{E+m}$, verify that

   $$\bar{u}u = -\bar{v}v = 2m\,,$$

   where $u = u(p,s)$ and $v = v(p,s)$ are positive and negative energy spinors respectively.

4. The charge conjugation operator $(C)$ takes a Dirac spinor $\psi$ into the "charge conjugate" spinor $\psi^C$, given by

   $$\psi^C = i\gamma^2\psi^*\,.$$

   Find the charge conjugates of $u^{(1)}$ and $u^{(2)}$, and compare them with $v^{(1)}$ and $v^{(2)}$.