Crash course on “applied QFT” (September 2023)
Homework No. 2 (Sep. 26, 2023)

1) Lorentz invariants

A Lorentz transformation is defined by

$$x_\mu \to x'_\mu = \Lambda^\nu_\mu x_\nu$$

with

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \det(\Lambda) = 1$$

where $v = \beta c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $c = 1$. How do the following quantities transform under this transformation?

1. The 3–volume element $d^3x \equiv dx\,dy\,dz$;
2. The 4–volume element $d^4x \equiv dt\,dx\,dy\,dz$;
3. $\delta(p^2 - m^2)$, where $p^2 \equiv p_\mu p^\mu$ and m is the mass;
4. The phase space element $d^3p/(2E)$, where $p = (E, \vec{p})$ is a momentum 4–vector and $d^3p \equiv dp_x\,dp_y\,dp_z$.

2) Noether’s theorem

Consider the following Lagrangian density with a complex scalar fields $\phi(x)$:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 |\phi|^2 \,.$$

1. Obtain the Euler–Lagrange equation of motion for the field ϕ.
2. Show that this Lagrangian is invariant under the transformation

   $$\phi(x) \to \mathrm{e}^{i\alpha} \phi(x) \,,$$

   where α is a real constant.
3. Invariance under this transformation means that this is a symmetry of the Lagrangian. Obtain the Noether current that corresponds to this symmetry. *Hint:* Consider the limit $|\alpha| \ll 1$.

4. Check that the Noether current is indeed conserved. *Hint:* Use the equations of motion.

3) Classical electromagnetic field equations

The classical electromagnetic field equations (with no sources) follow from the action

$$S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}),$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

1. Derive Maxwell's equations as the Euler–Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables. Write the equations in standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$. ($\epsilon^{ijk}$ is the totally antisymmetric rank-3 tensor.)

2. Construct the energy-momentum tensor for this theory:

$$\widehat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \, ,$$

where

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - \mathcal{L}g^{\mu\nu}, \quad K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu.$$

Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless ($\partial_\mu\partial_\lambda K^{\lambda\mu\nu} = 0$), so $\widehat{T}^{\mu\nu}$ is an equally good energy-momentum tensor with the same globally conserved energy and momentum. **Show** that this construction leads to an energy-momentum tensor $\widehat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\epsilon = \frac{1}{2}(E^2 + B^2); \quad \vec{S} = \vec{E} \times \vec{B}$$