Crash course on "applied QFT" (September 2023)
Homework No. 3 (Sep. 27, 2023)

1. Mandelstam variables

In a two-body scattering event, $A + B \to C + D$, it is convenient to introduce the *Mandelstam variables*

$$s = (p_A + p_B)^2\,;$$
$$t = (p_A - p_C)^2\,;$$
$$u = (p_A - p_D)^2\,.$$

The p's are energy–momentum four–vectors and the squares are Lorentz invariant ones, *e.g.* $p^2 = g_{\mu\nu}p^\mu p^\nu$. The *theoretical* virtue of the Mandelstam variables is that they are Lorentz invariants, with the same value in any inertial system. *Experimentally*, though, the more accessible parameters are energies and scattering angles.

(a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.

(b) Find the center–of–mass (CM) energy of C, in terms of s, t, u and the masses. (The CM frame is defined by $\vec{p}_A + \vec{p}_B = \vec{0}$.)

(c) Give the explicit expression for s in the rest frame of the "target" particle B (fixed–target scattering).

(d) Give the allowed ranges of t and u when the incoming momenta are fixed but the scattering angle is allowed to vary. *Hint:* Write t and u in terms of the momentum components defined in the CM frame.

2. Cross section for two-particle scattering

Let us consider the general two body process $1 + 2 \to 3 + 4$ ($2 \to 2$ scattering), where now the numbers label the momenta of the particles. The differential cross section is given by

$$d\sigma = \frac{1}{4E_1E_2|\vec{v}|}(2\pi)^4\delta^{(4)}(p_3 + p_4 - p_1 - p_2)\frac{d^3p_3}{2E_3(2\pi)^3}\frac{d^3p_4}{2E_4(2\pi)^3}|F|^2, \qquad (1)$$

where F is the reduced (Feynman) amplitude.

(a) Show that in both the CM and laboratory frames, the incident flux factor satisfies the equations

$$4E_1E_2|\vec{v}| = 4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2} = 2\lambda^{1/2}(s, m_1^2, m_2^2)\,, \tag{2}$$

where the kinematic function $\lambda(a,b,c) = (a-b-c)^2 - 4bc$. Note that in the rest frame of particle 2, $\vec{v}$ is the velocity of particle 1; in the CM frame, $\vec{v} = \vec{v}_1 - \vec{v}_2$.

(b) Perform the phase space integrals in Eq.(1) to get the differential cross section per unit solid angle in the CM frame:

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s \lambda^{1/2}(s, m_1^2, m_2^2)} \lambda^{1/2}(s, m_3^2, m_4^2)|F|^2\,, \tag{3}$$

where $d\Omega^* = d\phi^* d\cos\theta^*$ is the angular part of the phase space volume element, $d^3p^* = |\vec{p}^*|^2 d|\vec{p}^*| d\Omega^*$.

(c) Now express the same cross section in terms of the Mandelstam variable t, when there is cylindrical symmetry about the beam axis:

$$\frac{d\sigma}{dt} = \frac{|F|^2}{16\pi\lambda(s, m_1^2, m_2^2)}. \tag{4}$$

3. Product rules and trace theorems of $\gamma-$matrices

Prove the following properties of products of $\gamma-$matrices

$$\begin{aligned}
\gamma_\mu \gamma^\mu &= 4\,; \\
\gamma_\mu \not{a} \gamma^\mu &= -2\not{a}\,; \\
\gamma_\mu \not{a}\not{b} \gamma^\mu &= 4a \cdot b\,; \\
\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu &= -2\not{c}\not{b}\not{a}\,; \\
\mathrm{Tr}(\text{odd number of } \gamma' \text{s}) &= 0\,; \\
\mathrm{Tr}(\not{a}\not{b}) &= 4a \cdot b\,; \\
\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) &= 4[(a\cdot b)(c \cdot d) + (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d)]\,.
\end{aligned}$$

Hint: Use $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $(\gamma_5)^2 = 1$, $\{\gamma_5, \gamma^\mu\} = 0$ and the fact that the trace is cyclic. Here, γ_5 is defined as $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. For the first and third equations, a 4×4 unit matrix is understood to appear on the right–hand side.