Crash course on "applied QFT" (September 2023)
Homework No. 4 (Sep. 28, 2023)

1. **C, P and T transformation**
   In the chiral basis, the charge conjugation C, parity P and time reversal T on the Dirac field ψ are as follows (ignoring possible phase factors):[1]

$$\begin{aligned} C: &\ \psi(t,\vec{x}) \to -i\gamma^2\psi^*(t,\vec{x}),\\ P: &\ \psi(t,\vec{x}) \to \gamma^0\psi(t,-\vec{x}),\\ T: &\ \psi(t,\vec{x}) \to (-\gamma^1\gamma^3)\psi(-t,\vec{x}). \end{aligned}$$

   (a) Prove the explicit transformation properties for solutions of the free Dirac equation with positive and negative energy (u and v spinors; $s = \pm$ labels spin-up and spin-down, respectively):

$$\begin{aligned} C: &\ u(\vec{p},s) \to v(\vec{p},s), \quad v(\vec{p},s) \to u(\vec{p},s);\\ P: &\ u(\vec{p},s) \to u(-\vec{p},s), \quad v(\vec{p},s) \to -v(-\vec{p},s);\\ T: &\ u(\vec{p},s) \to u(-\vec{p},-s), \quad v(\vec{p},s) \to v(-\vec{p},-s). \end{aligned}$$

   (b) Show that the free Dirac Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

   is invariant under C,P and T separately.

   (c) Show that the components of the probability current $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ transform as expected for a 4–vector composed of a (charge) density $j^0 \equiv \rho$ and a (charge) 3–current $\vec{j}$.

2. **Spin Sums**

   Show that

$$\sum_{s=1}^{2} u(p,s)\bar{u}(p,s) = \not{p} + m, \ \sum_{s=1}^{2} v(p,s)\bar{v}(p,s) = \not{p} - m.$$

   Here s labels the two spin states. *Hint:* Use the Dirac representation of the γ matrices, and the associated solution of the Dirac equation shown in class. Note that, while $\bar{u}(p,s)u(p,s) = 2m$ is a 1–component object ("Dirac scalar"), $u(p,s)\bar{u}(p,s)$ is a 4×4 Dirac matrix!

[1] See e.g. Peskin and Schroeder, sec. 3.6.

3. **Scattering in Yukawa theory**

Let us consider the Yukawa theory described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}},$$

where

$$\mathcal{L}_{\text{free}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_\psi)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_\phi^2}{2}\phi^2, \quad \mathcal{L}_{\text{int}} = -\kappa\phi\bar{\psi}\psi.$$

Consider the $2 \to 2$ scattering of fermions (particles, not anti-particles):

$$\psi(\vec{k_1}, s_1) + \psi(\vec{k_2}, s_2) \to \psi(\vec{p_1}, s_1') + \psi(\vec{p_2}, s_2').$$

We work in the lowest non-trivial order in perturbation theory.

(a) Draw the Feynman diagrams.

(b) Write down the matrix element F using the Feynman rules given in class.

(c) Check that

$$F(\vec{k}_1, s_1; \vec{k}_2, s_2 \to \vec{p}_1, s_1'; \vec{p}_2, s_2') = -F(\vec{k}_1, s_1; \vec{k}_2, s_2 \to \vec{p}_2, s_2'; \vec{p}_1, s_1'),$$

as expected for an anti-symmetric wave function in the final (and initial) state.

(d) Perform the spin sum to compute the spin-averaged matrix-element squared

$$|\bar{F}|^2 = \frac{1}{4}\sum_{s_1 s_2 s_1' s_2'} |F|^2.$$

(e) Compute the differential cross section in the center of mass frame, $d\sigma/d\cos\theta^*$. Discuss the behavior at high energy, $s \gg m_\phi^2, m_\psi^2$.