Theoretical Particle Physics 1 (WS 23/24)
Homework No. 1 (Oct. 10, 2023)

1. **2–Body Decay**

   Consider the following Lagrangian:

$$\mathcal{L} = \bar{\psi}_1(i\partial\!\!\!/ - m_{f1})\psi_1 + \bar{\psi}_2(i\partial\!\!\!/ - m_{f2})\psi_2 + \frac{1}{2}(\partial^\mu\phi\partial_\mu\phi - m_\phi^2\phi^2) - g\phi(\bar{\psi}_2\psi_1 + \bar{\psi}_1\psi_2)\,. \quad (1)$$

   Here ψ_1 and ψ_2 are two Dirac fermions and ϕ is a real scalar; the coupling g is assumed to be real.

   (a) Can you think of a (discrete) symmetry that allows the interaction term written in eq.(1) but does not allow $\phi\bar{\psi}_1\psi_1$ and $\phi\bar{\psi}_2\psi_2$ terms?

   (b) Assuming $m_{f2} > m_{f1} + m_\phi$, which two–body decay is allowed in this theory? Draw the corresponding Feynman diagram, and write down the reduced Feynman amplitude.

   (c) Compute the squared amplitude, summed over spin in the final state and averaged over spin in the initial state, using the trace technique.

   (d) Compute the total two–body width by performing the two–body phase space integral explicitly.

2. **Classical Electrodynamics**

   In this problem we recover the classical description of electrodynamics from the manifestly Lorentz–covariant formalism presented in class.

   The covariant treatment started from the 4−vector

$$A^\mu = (V, \vec{A})\,. \quad (2)$$

   A gauge transformation could then be written as

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\chi(x)\,, \quad (3)$$

   where $\chi(x)$ is a differentiable real function of x. The field strength tensor is defined as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,. \quad (4)$$

   The Lagrange density in the absence of charges (i.e. for free fields) is then given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \quad (5)$$

   where repeated indices are summed.

(a) Write the gauge transformations (3) of the electric potential $V(\vec{x}, t)$ and the vector potential $\vec{A}(\vec{x}, t)$. *Hint:* Recall that $x^\mu = (t, \vec{x})$ implies $x_\mu = (t, -\vec{x})$, and $\partial^\mu = \partial/\partial x_\mu$.

(b) The electric field $\vec{E}$ and magnetic field $\vec{B}$ are defined by:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{6}$$

Using the results derived in the first part, show that $\vec{E}$ and $\vec{B}$ are gauge invariant.

(c) Using the definitions (4) and (6), show that the Lagrange density (5) can be written as

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right). \tag{7}$$

(d) From now on we assume that the scalar potential vanishes, $V = 0$. Show that $\vec{A} = A\vec{e}_x$, where $\vec{e}_x$ is a unit vector pointing in x direction, is a "pure gauge" if A is a constant, i.e. there exists a gauge transformation such that $A^\mu \to 0$. Find the corresponding function χ.

(e) Show that $\vec{E} = \vec{B} = 0$ if A^μ is a pure gauge. Moreover, show that for static fields, $\vec{A}$ is a pure gauge if $\vec{B} = 0$ *everywhere.*

(f) Finally, show that the integral of $\vec{A}$ over a closed contour,

$$\Phi = \oint \vec{A} \cdot d\vec{s}, \tag{8}$$

is gauge invariant. *Note:* This quantity appears in the description of the "Aharonov–Bohm effect" in quantum mechanics: the wave function of an electron changes, leading to changes in interference patterns, when the electron traverses a region with $\vec{B} = 0$ but $\vec{A} \neq 0$, if $\vec{B} \neq 0$ *somewhere inside* the closed loop of eq.(8).