Theoretical Particle Physics 1 (WS 23/24)
Homework No. 10 (December 18, 2023)
To be handed in by Sunday, January 7!

Quickies

About 20 to 25% of the points in the final exam will be awarded for "quickies", questions that you should be able to answer in a single sentence (or at most a short paragraph), or by drawing some diagram(s). From now on the homework sheets will contain examples of such questions. You might want to try answering these questions without looking at your notes (or a text book).

Q1: By looking at the relevant Feynman diagram(s), show that for $w \neq 1$, $P_{g \leftarrow q}(w) = P_{q \leftarrow q}(1-w)$; here w is the relative momentum fraction carried by the parton (quark or gluon) into which the original parton (here a quark) splits. Why does this not work at $w \to 1$?

Q2: In eq.(2.64) in class the concept of "fragmentation functions" $D_i^h(z, Q^2)$ was introduced, where i stands for a parton and h for a hadron; $z = E_h/E_i$ is the scaled energy variable, and Q describes the "hardness" (typical virtuality scale) of the process. Roughly, the fragmentation functions describe the relative flux of hadron h produced in the hadronization of a parton i. *(i)* How are $D_u^{\pi^+}$, $D_{\bar{d}}^{\pi^+}$ and $D_d^{\pi^-}$ related, when the mass difference between u and d quarks, as well as (small) contributions from electromagnetic interactions, are ignored? (In this limit "strong isospin" I is conserved, where u and d form an $SU(2)_I$ doublet.) *(ii)* Which constraints on sums of fragmentation functions follow from energy conservation?

Q3: List all the partonic QCD $2 \to 2$ scattering processes that have distinct matrix elements even in the limit of vanishing quark masses. *Hint:* There are eight of them!

1. Gauge Invariance in Gluon–Gluon Scattering

Consider the process $g(k_1)g(k_2) \to g(p_1)g(p_2)$. Let $\epsilon_1(k_1)$ and $\epsilon_2(k_2)$ be the polarization vectors of the gluons in the initial state, and $\epsilon_1'(p_1)$ and $\epsilon_2'(p_2)$ those of the gluons in the final state. The purpose of this exercise is to show that gauge invariance demands that the same coupling g_s appears in the three– and four–gluon vertices.

(a) Draw the Feynman diagrams that contribute. *Hint:* There are four of them.

(b) Write down the corresponding scattering amplitude (matrix element), using the Feynman rules (2.24) and (2.25) given in class for the three- and four gluon vertices, respectively.

(c) Gauge invariance requires that physical amplitudes vanish if the polarization vector of one of the external gluons is replaced by its 4−momentum. Show by explicit calculation that this is true when $\epsilon_1 \to k_1$ for the standard Feynman rules of QCD. *Hint:* Work in Feynman gauge, and use the fact that the *other* three gluons are transverse, $\epsilon_2 \cdot k_2 = \epsilon_1' \cdot p_1 = \epsilon_2' \cdot p_2 = 0$, see eq. (1.31). Using these relations, and the fact that the time components of the polarization vectors are zero, derive and use the relations $\epsilon_2 \cdot k_1 = \epsilon_1' \cdot p_2 = \epsilon_2' \cdot p_1 = 0$. Finally, you have to use the Jacobi identity $f_{abe} f_{cde} - f_{ace} f_{bde} + f_{ade} f_{bce} = 0$, see the second problem of HW 7.

2. Non–Singlet Quark Distribution Functions

A (flavor!) non–singlet quark distribution function is the *difference* between any two quark or antiquark distribution functions,

$$q_{\rm NS}(x, k^2) = q_i(x, k^2) - q_j(x, k^2) \,. \tag{1}$$

(The subscripts i, j have been suppressed on the left–hand side; it should be clear that many such differences can be defined.)

(a) Show that $q_{\rm NS}$ obeys a homogeneous evolution equation,

$$\frac{dq_{\rm NS}(x, k^2)}{d \ln k^2} = \frac{\alpha_S(k^2)}{2\pi} \int_x^1 \frac{dw}{w} P_{q \leftarrow q}(w) q_{\rm NS}\left(\frac{x}{w}, k^2\right) , \tag{2}$$

where $P_{q \leftarrow q}$ is the quark to quark splitting function introduced in eq.(2.55) in class.

(b) In order to solve eq.(2) perform a "Mellin transform", defined via

$$f(n) := \int_0^1 x^{n-1} f(x) dx \,. \tag{3}$$

Note that the function f may have additional arguments; in particular, it may depend on the squared momentum exchange k^2. Show that the Mellin transform turns the convolution of eq.(2) into a simple product,

$$\frac{dq_{\rm NS}(n, k^2)}{d \ln k^2} = \frac{\alpha_S(k^2)}{2\pi} P_{q \leftarrow q}(n) q_{\rm NS}(n, k^2) \,. \tag{4}$$

(c) Eq.(4) can now be solved by separation of variables. Using the explicit expression (2.34) for the running coupling $\alpha_S(k^2)$ (to 1–loop order), show that the solution can be written as

$$q_{\rm NS}(n, k^2) = q_{\rm NS}(n, k_0^2) \cdot \left(\frac{\alpha_S(k^2)}{\alpha_S(k_0^2)} \right)^{-\frac{6 P_{q \leftarrow q}(n)}{33 - 2N_f}} , \tag{5}$$

where N_f is the number of "active" quark flavor (with mass $m_q^2 < |k^2|$), and k_0 is some reference scale – like any first order differential equation, the solution of eq.(4) requires the specification of one boundary condition.

(d) Explicitly compute the first two moments ($n = 1, 2$) of $P_{q\leftarrow q}$ and interpret the result (5) for these cases.

Remark: From N quark and anti–quark distribution functions one can define $N - 1$ independent non–singlet distributions, plus one singlet distribution, $q_S = \sum_{i=1}^{N} q_i$; the evolution of this singlet distribution is coupled to that of the gluon density. This system of (only) 2 coupled equations is still much easier to solve than that of $N + 1$ coupled equations describing the original (anti–)quark densities plus the gluon density.