Theoretical Particle Physics 1 (WS 23/24)
Homework No. 11 (January 8, 2024)
To be handed in by Sunday, January 14!

Quickies

Q1: What limit does unitarity of the $S-$matrix impose on the behavior of total *exclusive* cross sections (e.g. $e^+e^- \to b\bar{b},\ e^+e^- \to W^+W^-$) at large Mandelstam$-s$?

Q2: What limit does unitarity of the $S-$matrix impose on the behavior of total *inclusive* cross sections (e.g. $pp \to b\bar{b} + X, pp \to t\bar{t} + X$) at large hadronic Mandelstam$-s$? *Hint:* Recall that a large hadronic Mandelstam$-s$ does *not* imply a large partonic Mandelstam$-s$!

Q3: Write down the propagator of one of the massive vector bosons in the IVB theory. (This is also the propagator of W bosons in the full Standard Model of particle physics, if the “unitary gauge” is chosen.)

Q4: Argue that the matrix element for the production of longitudinally polarized W bosons in e^+e^- annihilation via ν_e exchange in the $t-$channel, $e^+e^- \to W_L^+ W_L^-$, scales like s/m_W^2 at fixed scattering angle and $s \gg m_W^2$. To that end, show that the longitudinal polarization vector $\epsilon_0(k_w) \simeq k_W/m_W$ if energy $E_W \gg m_W$; here k_W is the 4–momentum of the W boson in question.

1. Muon decay

We consider the decay of muon $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ at the leading order of the $V - A$ interaction. Let p be the muon four momentum and let $k_i (i = 1, 2, 3)$ be the four–momenta of the final muon–neutrino, electron–antineutrino, and electron. Neutrinos can be treated as massless, and m_e^2 can be neglected ($m_\mu \simeq 206 m_e$).

(a) Draw the leading Feynman diagram and write down the corresponding amplitude.

(b) Compute the spin-averaged amplitude-squared, summed over spins of final particles as well. *Hint:* Recall the definition of $\bar{\Gamma} = \gamma_0 \Gamma^\dagger \gamma_0$ for the “Dirac conjugate” of some combination Γ of Dirac matrices. What is $\overline{\gamma_\mu \gamma_5}$? To evaluate the traces, you need the identities

$$\begin{aligned} \mathrm{tr}\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu &= 4i\epsilon_{\alpha\beta\mu\nu} \\ \epsilon_{\mu\lambda\alpha\beta}\epsilon^{\mu\rho\alpha\sigma} &= -2(\delta_\lambda^\rho\delta_\beta^\sigma - \delta_\beta^\rho\delta_\lambda^\sigma), \end{aligned} \qquad (1)$$

where $\epsilon_{\alpha\beta\mu\nu}$ is the (Lorentz invariant!) totally antisymmetric tensor in 4 dimensions, with $\epsilon_{0123} = -\epsilon^{0123} = 1$.

(c) Define

$$x_i = \frac{2(k_i \cdot p)}{m_\mu^2}. \tag{2}$$

(This is the ratio of the CM energy of particle i to the maximum available energy.) Show that

$$\sum_{i=1}^{3} x_i = 2 \tag{3}$$

and that the integral over the final state three–body phase space can be written as

$$\begin{aligned} \int d\Pi_3 &= \left[\int \prod_{i=1}^{3} \frac{d^3 k_i}{(2\pi)^3} \frac{1}{2E_i}\right] (2\pi)^4 \delta^{(4)}(p - \sum_i k_i) \\ &= \frac{m_\mu^2}{128\pi^3} \int dx_1 dx_2. \end{aligned} \tag{4}$$

What is the region of integration for x_1 and x_2? *Hint:* It is easier to work in the rest frame of the decaying muon. All Lorentz invariants involving only the final state momenta can be computed in terms of the x_i and the particle masses. To see that, dot the equation for 4–momentum conservation with the k_i in order to derive relations between the required product(s) $k_i \cdot k_j$ and x_1, x_2.

(d) Obtain the decay rate. What is the muon lifetime in seconds? *Hint:* Use $G_F = 1.15 \cdot 10^{-5}$ GeV^{-2} for the Fermi constant.

Reminder: If you wish to participate in the final exam, please register with Basis until January 15!