Theoretical Particle Physics 1 (WS 23/24)

Homework No. 2 (Oct. 16, 2023)

To be handed in by Sunday, October 22!

1. **Current conservation in QED**

   The QED Lagrangian reads [see eq. (1.16) in class]:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_k \bar{\psi}_k(i\partial\!\!\!/ - m_k - q_k A\!\!\!/)\psi_k. \tag{1}$$

   It is invariant under transformations of the type [see eq. (1.12) in class]

$$\psi_k \to e^{i\alpha q_k}\psi_k \,,$$

   where α is a real constant. The corresponding Noether current is [see eq. (1.13) in class]

$$-J_N^\mu \equiv J_Q^\mu = \sum_k q_k \bar{\psi}_k \gamma^\mu \psi_k. \tag{2}$$

   (a) Write down the equations of motion for fermion fields ψ_k and their conjugates $\bar{\psi}_k$ using the Lagrangian (1).

   (b) Using these equations of motion, show that the current J_Q^μ of eq. (2) is conserved, $\partial_\mu J_Q^\mu = 0$.

   (c) Is the sum over k in eq. (2) necessary for the current to be conserved? What does this mean?

2. **Lorentz Transformation of Spinors**

   In class it was shown that the generators of Lorentz transformations of spinors are given by

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu \,, \gamma^\nu\right] \,. \tag{3}$$

   (a) Show by explicit calculation that the $S^{\mu\nu}$ satisfy the Lorentz algebra, i.e.

$$\left[S^{\mu\nu} \,, S^{\alpha\beta}\right] = i\left(S^{\nu\alpha}g^{\mu\beta} + S^{\mu\beta}g^{\nu\alpha} - S^{\mu\alpha}g^{\nu\beta} - S^{\nu\beta}g^{\mu\alpha}\right) \,. \tag{4}$$

   Hint: Use the Dirac algebra, i.e. the anticommutator relation $\{\gamma^\rho \,, \gamma^\sigma\} = 2g^{\rho\sigma}$. To that end, write out the l.h.s. of eq.(4) explicitly using the definition given in eq.(3), and begin by anticommuting the two middle γ matrices in each of the 8 terms.

(b) Write down the explicit expressions for the S^{0k} and S^{lk} in the Dirac representation, the equivalent of eq.(L.14) in class for the chiral representation, and show that it also satisfies

$$(S^{\mu\nu})^\dagger \gamma^0 = \gamma^0 S^{\mu\nu} \,. \tag{5}$$

Hence eq.(L.19) shown in class also holds in this representation.

(c) The generators S with two spatial indices, S^{kl}, generate rotations. In order to see this, recall from (L.17b) that

$$\Lambda_{\frac{1}{2}}(\omega_{\mu\nu}) = \exp\left[-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right] , \tag{6}$$

where the six components of $\omega_{\mu\nu}$ contain three rotation angles and three boost parameters. Specifically, consider

$$\omega_{12} = -\omega_{21} = \theta_3 \neq 0 \,, \tag{7}$$

with all other $\omega_{\mu\nu} = 0$, and show that this describes a rotation around the z axis of the two 2–spinors forming a Dirac spinor in the Dirac representation.

(d) Finally, consider a Lorentz transformation with $\omega_{01} = -\omega_{10} = 2\eta_1$, with all other $\omega_{\mu\nu} = 0$. Define

$$\Sigma_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} , \tag{8}$$

and show that $\Sigma_1^2 = 1$. Use this to show that

$$\Lambda_{\frac{1}{2}}(\eta_1) = \cosh(\eta_1) + \Sigma_1 \sinh(\eta_1) \,.$$

Finally, show that $\Lambda_{\frac{1}{2}} u(\vec{p} = 0)$ gives the $u-$spinor for a fermion moving in $x-$direction, with $\cosh(\eta_1) = \sqrt{(E+m)/(2m)}$.