Theoretical Particle Physics 1 (WS 23/24)

Homework No. 3 (Oct. 23, 2023)

To be handed in by Sunday, Oct. 29!

1. **“Gauge Fixing”**

The QED Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_k \bar{\psi}_k (i\partial\!\!\!/ - m_k - q_k A\!\!\!/) \psi_k \,. \tag{1}$$

Now, let us add the following “gauge fixing” term:

$$\mathcal{L}_\xi = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2 = -\frac{1}{2\xi} (\partial_\mu A^\mu)(\partial_\nu A^\nu) \tag{2}$$

to the QED Lagrangian (1), so that the total Lagrangian becomes

$$\mathcal{L}_{tot} = \mathcal{L}_{QED} + \mathcal{L}_\xi \,. \tag{3}$$

This is legitimate at least in Lorenz gauge, where $\partial_\mu A^\mu = 0$. ξ is an arbitrary dimensionless constant.

(a) Write down the equation of motion for the photon field A_μ that follows from the “gauge fixed” Lagrangian (3).

(b) Use this equation of motion to show that the photon propagator becomes

$$D^{\mu\nu}(k) = \frac{-i}{k^2} \left[g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right] , \tag{4}$$

where k is the 4−momentum flowing through the propagator. The case discussed in class corresponds to the “Feynman gauge”, $\xi = 1$. [*Hint:* You may employ the “iterated first order” formalism used in class, *without* applying the Lorenz gauge condition $k \cdot A = 0$.]

(c) Show that the extra terms $\propto k^\mu k^\nu$ do not change the (leading order) amplitudes for scattering processes. To that end, show that this extra term vanishes when it is multiplied with the current $\bar{u}\gamma_\nu u$ (t - channel) or $\bar{u}\gamma_\nu v$ (s - channel). (The Feynman rules of QED imply that every photon propagator must be multiplied with such currents.) In other words, scattering amplitudes do not depend on the (arbitrary, unphysical) parameter ξ. [*Hint:* Use 4-momentum conservation at the vertex, taking care to distinguish between particles and antiparticles, and use the (free) equations of motion for the external fermions.]

(d) How is this result related to current conservation?

2. Scalar QED

The Lagrangian for a free complex scalar field ϕ reads:

$$\mathcal{L}_{\text{free }\phi} = \partial_\mu \phi^* \partial^\mu \phi - m^2 |\phi|^2 \,. \tag{5}$$

It is invariant under a global transformation

$$\phi \to e^{i\alpha q} \phi \,. \tag{6}$$

(a) Show that the corresponding Noether current can be written as

$$J_N^\mu = q(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) \,, \tag{7}$$

which is proportional to the probability current of this (Klein–Gordon) theory. *Hint:* Assume $|\alpha| \ll 1$, and remember that ϕ and ϕ^* can be treated as independent degrees of freedom.

(b) The Lagrangian (5) is not invariant under *local* transformations of the type (6), where α depends on x. Show that an invariant Lagrangian can be constructed by replacing the ordinary derivative ∂_μ by the gauge covariant derivative [see eq. (1.18) in class]

$$D_\mu = \partial_\mu + iqA_\mu \,, \tag{8}$$

if the photon field transforms as usual, $A_\mu \to A_\mu - \partial_\mu \alpha(x)$ [see eq. (1.17) in class].

(c) Derive the Feynman rules that correspond to the $\phi^* \phi A_\mu$ and $\phi^* \phi A_\mu A_\nu$ interactions in the Lagrangian obtained in the previous step. Where does the current (7) appear? *Hint:* Recall that the vertex factors can be obtained by taking (functional) derivatives of the Lagrangian.

(d) Write down the lowest order amplitude for $\gamma\gamma \to \phi^+ \phi^-$ (i.e. a charged scalar pair). Check that the amplitude vanishes if one of the polarization vectors is replaced by that photon's 4−momentum; recall that this is required by QED gauge invariance.

Note: Scalar QED plays a crucial role in supersymmetric extensions of the Standard Model of particle physics, where ϕ can e.g. stand for the scalar "superpartner" of a charged lepton.