Theoretical Particle Physics 1 (WS 23/24)

Homework No. 5 (Nov. 12, 2023)

To be handed in by November 19, 2023!

1. Proton Radius

In this exercise we will show that the charge radius of the proton can be determined from its electric form factor, which had been discussed in class.

(a) Show that in the non–relativistic limit, the energy transfer $|k_0|$ is much smaller than the 3–momentum transfer $|\vec{k}|$; we can then write $k \simeq (0, \vec{k})$ with $k^2 = -\vec{k}^2$.

(b) In this limit, the electric form factor is the (three–dimensional) Fourier transform of the charge distribution:

$$G_E(k^2) = \int d^3x \, \rho_Q(\vec{x}) e^{i\vec{k}\cdot\vec{x}} \,. \tag{1}$$

Show that this implies

$$\left.\frac{dG_E(k^2)}{dk^2}\right|_{k^2 \to 0} = \frac{2\pi}{3} \int dr \, r^4 \, \rho_Q(r) \,. \tag{2}$$

Hint: Use spherical coordinates with $|\vec{x}| = r$ and assume that the charge density ρ_Q is spherically symmetric, i.e. $\rho_Q(\vec{x}) = \rho_Q(r)$. Perform the angular integrals, and expand the resulting trigonometric functions including terms up to $\mathcal{O}(k^3 r^3)$.

(c) Show that the right–hand side of eq.(2) is proportional to the square of the proton charge radius:

$$r_p^2 = \frac{\int d^3x \, |\vec{x}|^2 \rho_Q(\vec{x})}{\int d^3x \, \rho_Q(\vec{x})} \,. \tag{3}$$

(d) The same formalism can also be used to constrain the radius of the electron or muon. We saw that the proton form factors can be described by an ansatz $G_E(k^2) \propto (1 - k^2/\Lambda^2)^{-2}$, where Λ is some energy or mass scale. Show that for $|k^2| \ll \Lambda^2$ this ansatz leads to a point interaction term (e.g. coupling four fermions seemingly without propagator) in $G_E(k^2)/k^2$, where the $1/k^2$ comes from the photon propagator. How is Λ related to the radius of the particle in question?

2. Feynman parameter integrals

(a) Prove the following identities by explicit calculation:

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \frac{1}{(Ax+By)^2}\delta(x+y-1)\,; \tag{4}$$

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{2}{(Ax+By+Cz)^3}\delta(x+y+z-1)\,. \tag{5}$$

Hint: Start with the simpler identity, and use it in the derivation of the more complicated result. Remember that after integrating over the $\delta-$function, the sum over the integration variables must be ≤ 1!

(b) Use the first identity (4) to prove by induction in n that

$$\frac{1}{A^n B} = \int_0^1 dx \int_0^1 dy \, \frac{n x^{n-1}}{(Ax+By)^{n+1}}\delta(x+y-1) \tag{6}$$

Note: These identities are used in the calculation of loop diagrams, where A, B, C are generally functions of various kinematic Lorentz invariants. However, in order to derive the above equations it is sufficient to consider A, B, C to be constants (i.e., independent of the "Feynman parameters" x, y, z).