Theoretical Particle Physics 1 (WS 23/24)
Homework No. 6 (Nov. 17, 2023)
To be handed in by Sunday, November 26, 2023!

1. Wick rotation

When calculating radiative (loop) corrections in class, we encounter integrals of the type

$$I(\Delta, n) = \int_{-\infty}^{\infty} \frac{d^4Q}{(2\pi)^4} \frac{1}{(Q^2 - \Delta)^n}, \tag{1}$$

where Δ does not depend on the 4–momentum Q. Recall that $Q^2 = (Q^0)^2 - (\vec{Q})^2$, i.e. the integral in (1) has to be performed in Minkowski space. It is much more convenient to perform such integrals in 4–dimensional Euclidean space. To that end, one introduces a (complex) Euclidean 4–momentum,

$$Q^0 \equiv iQ_E^0; \; \vec{Q} \equiv \vec{Q}_E. \tag{2}$$

(a) Show that (1) becomes

$$I(\Delta, n) = \frac{i(-1)^n}{(2\pi)^4} \int_0^{\infty} d^4Q_E \frac{1}{(Q_E^2 + \Delta)^n}, \tag{3}$$

where as usual in Euclidean space, $Q_E^2 = (Q_E^0)^2 + (\vec{Q}_E)^2$.

(b) To make further progress, rewrite the integral in (3) into an integral over 4–dimensional angular variables and the absolute value $|Q_E|$, $\int d^4Q_E = \int d\Omega_4 |Q_E|^3 d|Q_E|$. *Hint:* Use 4–dimensional spherical coordinates, where

$$Q_E = |Q_E|(\sin\omega \sin\phi \sin\theta, \cos\omega \sin\phi \sin\theta, \cos\phi \sin\theta, \cos\theta),$$

so that $d^4Q_E = |Q_E|^3 \sin^2\theta \sin\phi d\omega d\phi d\theta d|Q_E|$, to show that the angular integral $\int d\Omega_4 = 2\pi^2$.

(c) Finally perform the integral over $|Q_E|$ to show that for $n > 2$,

$$I(\Delta, n) = \frac{i(-1)^n}{16\pi^2} \frac{\Delta^{2-n}}{(n-1)(n-2)}. \tag{4}$$

2. $g-2$ of the Electron

In class we had derived the following expression for the vertex correction in QED, see eqs.(1.93), (1.94):

$$I^\mu = 2e^3\bar{u}(p_2)\int\frac{d^4Q}{(2\pi)^4}\int_0^1 dx\int_0^{1-x} dy\left[2mP^\mu\left(x+y-(x+y)^2\right)+\gamma^\mu(\ldots)\right]/D^3 u(p_1)\,, \tag{5}$$

with $P = p_1 + p_2$ and

$$D = Q^2 - m^2(x+y)^2 + xyk^2\,, \tag{6}$$

$k = p_2 - p_1$ being the 4–momentum of the photon. This result can be used to derive the first order correction to the magnetic moment of the electron (or muon). To that end, you can proceed as follows:

(a) Write

$$I^\mu = ie\bar{u}(p_2)\left(\gamma^\mu F_1 + i\frac{\sigma^{\mu\nu}k_\nu}{2m}F_2\right)u(p_1).$$

What is F_2 in our case? *Hint:* Use the Gordon decomposition, eq.(3.66) in class!

(b) The dipole moment is measured in a macroscopic external magnetic field. This corresponds to an interaction with "static photons", which have $k^2 = 0$. (You should not set $k = 0$ in the numerator, since the k in front of F_2 corresponds to taking derivatives of the vector potential in coordinate space. These derivatives are required to get $\vec{B}$.)

(c) The dipole moment is then simply given by

$$g_e = 2 + 2F_2(0).$$

It can be evaluated using the result (4) of the previous problem.

Note: g_e has now been evaluated to 5–loop order. The prediction agrees with experiment to 7 parts in 10^{10}. This is probably the most precise test of any physical theory.