Theoretical Particle Physics 1 (WS 23/24)
Homework No. 7 (Nov. 27, 2023)
To be handed in by Sunday, December 3, 2023!

1. Vacuum Polarization in QED

In class, eq. (1.98), it has been shown that the vacuum polarization (correction to the photon propagator) is described by the function

$$\Pi^{\mu\nu}(k^2) = 4ie^2 \int \frac{d^4Q}{(2\pi)^4} \int_0^1 dx \frac{2Q^\mu Q^\nu - g^{\mu\nu}Q^2 + g^{\mu\nu}k^2x(1-x) + g^{\mu\nu}m^2}{[Q^2 + k^2x(1-x) - m^2]^2}, \quad (1)$$

where k is the 4−momentum of the (virtual) photon, and Q the loop momentum.

(a) Perform the "Wick rotation" as described in homework 6, eq. (2). *Hint:* Use $Q^\mu Q^\nu \to Q^2 g^{\mu\nu}/d$, which is valid since the denominator in (1) depends on Q only through Q^2, and recall that $Q^2 = -Q_E^2$. (Here d is the number of spacetime dimensions.)

(b) Evidently the integral over $|Q_E|$ is badly divergent. This is regularized by performing the calculation in $d < 4$ dimensions. The results of problem 1 of homework 6, therefore, have to be generalized to

$$\int \frac{d^dQ_E}{(2\pi)^d} \frac{1}{(Q_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \Delta^{\frac{d}{2}-n}; \quad (2)$$

$$\int \frac{d^dQ_E}{(2\pi)^d} \frac{Q_E^2}{(Q_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - 1 - \frac{d}{2})}{\Gamma(n)} \Delta^{\frac{d}{2}+1-n}, \quad (3)$$

where $\Gamma(t) = \int_0^\infty \exp(-y) y^{t-1} dy$ is the Gamma function; for positive integer argument ℓ, it is given by $\Gamma(\ell) = (\ell - 1)!$. Check that these identities agree with eq. (4) of homework 6, for convergent integrals (sufficiently large n).

(c) Use the identities (2), (3) to perform the (Wick-rotated) loop integral in (1). The result can now be written as

$$\Pi^{\mu\nu}(k^2) = k^2 g^{\mu\nu} \Pi(k^2), \quad (4)$$

for some (scalar) function $\Pi(k^2)$.

(d) Combine the two terms in the integral of x, using

$$t\Gamma(t) = \Gamma(t + 1) \quad (5)$$

to show that the result becomes proportional to $\Gamma(2 - \frac{d}{2})$.

(e) Take the limit $\epsilon = 4 - d \to 0$ neglecting the term of $O(\epsilon)$, using

$$\Gamma(2 - \frac{d}{2}) = \frac{2}{\epsilon} - \gamma_E + O(\epsilon), \tag{6}$$

where γ_E is a constant. (The result is still divergent.)

(f) We are interested in the *difference*: $\hat{\Pi}(k^2) = \Pi(k^2) - \Pi(0)$, which will correspond to the effective electric charge

$$\alpha_{eff}(k^2) = \frac{\alpha^2}{1 - \hat{\Pi}(k^2)}. \tag{7}$$

Discuss the high energy limit $|k^2| \gg m^2$.

2. Non–Abelian gauge symmetry and the Adjoint representation

In this problem we discuss some issues in group theory, more exactly the representation theory of Lie groups.

(a) Show that the *Jacobi* identity

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$$

is fulfilled for arbitrary X_a ($[A, B] = AB - BA$ is the commutator).

(b) A Lie algebra is defined via the following relations for its generators:

$$[T_a, T_b] = i f^{abc} T_c\,, \qquad \mathrm{tr}(\mathrm{T_a T_b}) = \frac{1}{2}\delta_{\mathrm{ab}}\,. \tag{8}$$

The f^{abc} are called *structure constants*. Show that they are totally antisymmetric in all indices. *Hint:* To prove the antisymmetry under $b \leftrightarrow c$, multiply the first eq.(8) with T_d and take the trace; recall that a trace of a product of matrices is invariant under cyclical permutations of these matrices.

(c) Show that the matrices $(T_a)^{bc} := -i f^{abc}$ satisfy the first eq.(8). This representation of the algebra is called the *adjoint representation*. (It does not satisfy the second eq.(8), which defines the normalization.) *Hint:* Use the Jacobi identity and the definition (8) to derive an identity for a sum (with three terms) of products of two structure constants.