Theoretical Particle Physics 1 (WS 23/24)
Homework No. 8 (Dec. 3, 2023)
To be handed in by Sunday, December 10, 2023!

Quickies

About 20 to 25% of the points in the final exam will be awarded for "quickies", questions that you should be able to answer in a single sentence (or at most a short paragraph), or by drawing some diagram(s). From now on the homework sheets will contain examples of such questions. You might want to try answering these questions without looking at your notes (or a text book).

Q1: How does the coupling strength of QED change when the magnitude of the virtuality of the photon coupling to the relevant vertex is increased (or, a little more sloppily, at higher energy)?

Q2: Draw all the diagrams (if any) contributing to: *(i)* $e^+e^- \to e^+e^-$ (Bhabha scattering); *(ii)* $e^-e^- \to e^-e^-$ (Møller scattering); *(iii)* $e^-e^- \to e^+e^+$.

Q3: Bhabha scattering *at small (forward) scattering angle* is often used to determine the luminosity of e^+e^- colliders. Give two properties which make this reaction ideally suited for that purpose.

Q4: How many generators does the QCD gauge group $SU(3)_C$ have?

1. Non–Abelian gauge symmetry: Lagrangian

Let us take a free Dirac field Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)\,. \tag{1}$$

Here ψ stands for an object with N components, i.e. it carries an index k that runs from 1 to N. Note that all N components have the same mass, i.e. we are dealing with N degenerate states.

(a) The Lagrangian $\mathcal{L}_0$ is not invariant under local $SU(N)$ transformations:

$$\psi \to \psi' = U(x)\psi,\ U(x) = \exp\left(ig\sum_a \chi_a(x)T_a\right),\ U(x)^\dagger U(x) = 1\,. \tag{2}$$

(Evidently, U is a unitary $N\times N$ matrix.) Therefore, we want to *gauge* the symmetry: We introduce a (gauge) covariant derivative by minimal coupling to a gauge field and identify the gauge field's transformation properties. The covariant derivative is defined by the requirement that $D_\mu\psi$ transforms in the same way as ψ itself:

$$D_\mu\psi = (\partial_\mu + igA_{\mu a}T_a)\psi\,, \tag{3}$$

and demand

$$(D_\mu\psi) \to (D_\mu\psi)' = U(x)(D_\mu\psi)\,. \tag{4}$$

Show that for small gauge transformations, $|\chi_a| \ll 1\ \forall a$, this is equivalent to demanding that the gauge boson transformations as

$$A_{\mu a} \to A'_{\mu a} = A_{\mu a} - g f_{abc}\chi_b A_{\mu c} - \partial_\mu \chi_a\,. \tag{5}$$

Hint: Expand the exponential at the appropriate place in the calculation.

(b) Show that

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) \tag{6}$$

is now gauge invariant.

(c) Define the field strength tensor by

$$(D_\mu D_\nu - D_\nu D_\mu)\psi =: ig(F^a_{\mu\nu}T^a)\psi. \tag{7}$$

Show that this gives

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu\,. \tag{8}$$

(d) Note that the covariant derivative was constructed such that $D'_\mu U(x) = U(x)D_\mu$ holds. Therefore

$$[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = U(x)(D_\mu D_\nu - D_\nu D_\mu)\psi\,. \tag{9}$$

Use this to derive the transformation property of the field strength tensor,

$$F_{\mu\nu} := F^a_{\mu\nu}T^a \to F'_{\mu\nu} = U F_{\mu\nu} U^{-1}\,. \tag{10}$$

Show that for infinitesimal transformation, $|\chi^a| \ll 1$, this reduces to the transformation law given in class, eq.(2.20):

$$F^a_{\mu\nu} \to F^{a\prime}_{\mu\nu} = F^a_{\mu\nu} - g f^{abc}\chi^b F^c_{\mu\nu}\,. \tag{11}$$

(e) Because of the last equation, the field strength tensor itself is not gauge invariant. Verify that the expression

$$\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{12}$$

is invariant. The trace is taken over the matrix entries of the generators. As this term is gauge invariant and has the proper units, we can add it to the Lagrangian. It gives rise to self couplings of the gauge bosons. The final result for the gauge invariant Dirac Lagrangian (for one species of fermion) is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) - \frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\,. \tag{13}$$

(f) The generators of the algebra T^a are normalized to $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. Show that, in the case of $SU(2)$, the normalization condition is fulfilled by the $T^a = \frac{1}{2}\sigma^a$, where $\sigma^a,\ a = 1,2,3$ are the Pauli matrices.

(g) Prove that

$$\frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{4}F_{a\mu\nu}F^{\mu\nu a}\,. \qquad (14)$$

2. Gauge (non-)invariance of hadronic wave functions

(a) Show that the "baryon" wave function $\sum_{i,j,k=1}^{3} \epsilon_{ijk} q_i q_j q_k$ is invariant under an infinitesimal $SU(3)$ gauge transformation, where ϵ_{ijk} is the completely antisymmetric tensor of rank 3, and $i, j, k \in \{1, 2, 3\}$ are color indices.

(b) Show that the "meson" wave function $\sum_{i=1}^{3} \bar{q}_i q_i$ is also invariant under an infinitesimal $SU(3)$ gauge transformation.

(c) Show that the wave function $\bar{q}_1 q_1$ is *not* invariant under a general infinitesimal $SU(3)$ transformation.