Theoretical Particle Physics 1 (WS 23/24)

Homework No. 9 (Dec. 11, 2020)

To be handed in by Sunday, December 17!

Quickies

About 20 to 25% of the points in the final exam will be awarded for "quickies", questions that you should be able to answer in a single sentence (or at most a short paragraph), or by drawing some diagram(s). From now on the homework sheets will contain examples of such questions. You might want to try answering these questions without looking at your notes (or a text book).

Q1: How does the coupling strength of QCD (with at most 6 "active" quark flavors, $N_f \leq 6$) change when the magnitude of the virtuality of the gluon coupling to the relevant vertex is increased (or, a little more sloppily, at higher energy)? What happens for $N_f \geq 17$?

Q2: Consider deep–inelastic electron proton scattering, $ep \to eX$. How much does scattering on down quarks contribute to the double differential cross section, $d^2\sigma(ep \to eX)/(dx\, dk^2)$, relative to scattering on up quarks? Here x is the Bjorken variable and k the 4−momentum exchange.

Q3: Why can the cross section for strong processes only be computed reliably in perturbation theory if the process is defined sufficiently inclusively, e.g. after summing over *all* hadronic final states in deep inelastic scattering?

Q4: Draw all tree–level Feynman diagrams (if any) that contribute to: *(i)* $e^+e^- \to q\bar{q}g$, where q is a quark (of specific flavor), and g is a gluon; *(ii)* $e^-e^- \to qqg$; *(iii)* $qq \to gg$; *(iv)* $q\bar{q} \to gg$.

1. Brute-force computation in $SU(3)$

The standard basis for the generators of $SU(3)$ (for the fundamental representation) is

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Note: $T^a = \lambda^a/2$, where the λ^a are the Gell-Mann matrices.

(a) Show that T^1, T^2 and T^3 form an $SU(2)$ algebra. Which generator commutes with these three?

(b) Check the orthogonality condition

$$\mathrm{tr}[T^a T^b] = C\delta^{ab}, \tag{1}$$

and evaluate the constant C for this representation.

(c) Compute a constant C_2, called the quadratic Casimir operator, from its definition

$$\sum_a T^a T^a = C_2 \mathbf{1}. \tag{2}$$

Verify the relation

$$d(r)C_2 = d(G)C, \tag{3}$$

where $d(r)$ is the dimension of the representation being considered and $d(G)$ is that of adjoint.

2. Quark-Antiquark annihilation

Consider the QCD process $\bar{u}_i u_j \to \bar{d}_k d_l$, where u and d stands for an up- and down-quark, and $i, j, k, l \in \{1, 2, 3\}$ are color indices. In this exercise we are only interested in the color structure of the matrix element. (The rest of the calculation obviously proceeds in essentially the same way as for $e^+e^- \to \mu^+\mu^-$.)

(a) Write down the amplitude, including the color factor. *Hint:* Remember to sum over the color index of the virtual (exchanged) gluon!

(b) Compute the square of the color factor (products of elements of the generator matrices T_a) that appears in the squared amplitude, for *fixed* indices i, j, k, l of the external quarks.

(c) For physical situations, one has to sum over the color in the final state. Therefore perform the sum over k and l, using $\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$, see eq. (1).

(d) Using the explicit representations of the generators T_a given in problem 1, compute the factor obtained in step 3 for all 9 combinations of i and j; this is one way to compute the final color factor, which is the color average $\frac{1}{9}\sum_{i,j=1}^{3}$ of the color factor of step 3. Interpret the result.