String Theory

Winter Term 2008/2009

Problem Sheet 2

Discussion: October 29, 14:00 in Hörsaal 118, AVZ

1. Energy-momentum tensor

   The action for a field theory is given by the integral of the Lagrangean, $S = \int \mathrm{d}^4x \sqrt{-g}\mathscr{L}$. For a real scalar field the action is

   $$S_\phi = \int \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \right) ,$$

   while for an electromagnetic field with field A_μ with field strength $F_{\mu\nu}$ it is

   $$S_{\mathrm{em}} = \int \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right) .$$

   (a) Calculate the energy-momentum tensors using the "general relativity prescription"

   $$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} .$$

   (b) For a flat spacetime, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$, the action is invariant under infinitesimal translations $x^\mu \to x^\mu + a^\mu$. This symetry entails a conserved Noether current (the "canonical" energy-momentum tensor),

   $$T^{\mathrm{can}}_{\mu\nu} = -\eta_{\mu\rho}\frac{\partial\mathscr{L}}{\partial\left(\partial_\rho\phi\right)}\partial_\nu\phi + \eta_{\mu\nu}\mathscr{L} .$$

   Determine this tensor for both actions! Show that

   i. $T_{\mu\nu}$ and $T^{\mathrm{can}}_{\mu\nu}$ coincide for the scalar field, but

   ii. they don't for the electromagneitc field. Show that the mismatch is a totla derivative, i.e. that $T_{\mu\nu} = T^{\mathrm{kan}}_{\mu\nu} + \partial_\rho\left(\ldots\right)^\rho_{\mu\nu}$.

2. Consider a field theory with some fields ϕ_i coupled to a metric $g_{\mu\nu}$. Let the action $S[g_{\mu\nu}, \phi_i]$ be invariant under arbitrary rescalings of the metric, $g_{\mu\nu} \to \lambda^2(x)\,g_{\mu\nu}$, where $\lambda \neq 0$. The fields ϕ_i may transform in an arbitrary way under this rescaling.

   Show that the energy-momentum tensor

   $$T_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}S[g_{\mu\nu}, \phi_i]$$

   is traceless!

3. Show that the induced metric on the worldvolume of any p-brane is

$$G_{ab} = \partial_a x^\mu \partial_b x^\nu g_{\mu\nu} \,,$$

where $g_{\mu\nu}$ is the spacetime metric and the worldvolume is parametrised by some coordinates σ^a, $a = 0, \ldots, p$!

4. Check that the Nambu–Goto and Polyakov actions are invariant under reparametrisations $\sigma^a \to \sigma^{a\prime}(\sigma)$!

5. Show that in two dimensions, the Riemann tensor is completely determined by the Ricci scalar (by using the symmetries of the Riemann tensor)! Derive the explict relation!