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Loday-Type Algebras and the Rota–Baxter Relation

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Abstract. In this brief Letter, we would like to report on an observation concerning the relation between Rota–Baxter operators and Loday-type algebras, i.e. dendriform di- and trialgebras. It is shown that associative algebras equipped with a Rota–Baxter operator of arbitrary weight always give such dendriform structures.

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1. Introduction

Rota–Baxter operators* of weight $\lambda \in \mathbb{K}$ fulfil the so-called Rota–Baxter relation which may be regarded as one possible generalization of the standard shuffle relation [1, 2]. They appeared for the first time in the work of the mathematician G. Baxter [3] and were then intensively studied by F. V. Atkinson [4], J. B. Miller [5], G.-C. Rota [6, 7], P. Cartier [8] and others, while more recently they reappeared in the work of L. Guo [2].

There is a lot more to say about Rota–Baxter operators and the Rota–Baxter relation which will be part of a series of future work devoted to certain other aspects of Rota–Baxter operators and the Rota–Baxter relation. Especially the class of idempotent Rota–Baxter operators showed up to be of importance recently with regards to the Hopf algebraic background of renormalization, in particular in the context of the minimal subtraction scheme and the Riemann–Hilbert problem [9, 10].

By Loday-type algebras we mean dendriform di- and trialgebras [11, 12]. Such algebras are equipped with two respectively three algebra compositions fulfilling certain relations.

In an interesting and inspiring work M. Aguiar [13] showed, beside other results, that the class of Rota–Baxter operators of weight $\lambda = 0$ defined on an associative $\mathbb{K}$-algebra $\mathcal{A}$ allows one to define a dendriform dialgebra due to the Rota–Baxter relation which reduces in the case of zero weight ($\lambda = 0$) to a mere shuffle relation.

*Here we call them Rota–Baxter operators, whereas in the literature they are generally called Baxter operators. This serves mainly to distinguish them clearly from the Yang–Baxter family of objects—by the way, both Baxters are different.

Here we would like to show that one can extend the aforementioned result of Aguiar to an associative $\mathbb{K}$-algebra equipped with a general Rota–Baxter operator of weight $\lambda \in \mathbb{K}$ fulfilling the Rota–Baxter relation. Let us call such an associative $\mathbb{K}$-algebra containing a Rota–Baxter operator a Rota–Baxter algebra. We will see that the most natural dendriform structure on a Rota–Baxter algebra is a dendriform trialgebra one. The connection between Rota–Baxter algebras and Loday-type algebras provides a rich class of examples for the latter. Our observation was inspired by a paper of M. A. Semenov-Tian-Shansky on the classical r-matrix and the so-called modified classical Yang–Baxter equation defined therein [15].

This paper is organized as follows. In the next section, we first give a brief sketch of the concept of Rota–Baxter algebras mainly relying on the articles by L. Guo [2] and G.-C. Rota [1]. Section three contains the definitions of dendriform di- and trialgebras as they can be found in the exhaustive work of J.-L. Loday and collaborators [11, 12]. In Section 4, we extend the observation of M. Aguiar [13] to general Rota–Baxter operators including the link to dendriform trialgebras. This provides a whole new class of examples for these algebraic structures. The last section ends with a short summary and an outlook to the forthcoming work in progress.

2. The Rota–Baxter Relation

Let $\mathcal{A}$ be an associative $\mathbb{K}$-algebra. $\mathbb{K}$ is supposed to be a field ($\mathbb{C}$ or $\mathbb{R}$). The linear operator R: $\mathcal{A} \to \mathcal{A}$ must fulfil the following relation:

$$R(x)R(y) + \lambda R(xy) = R(R(x)y + xR(y)), \quad x, y \in \mathcal{A}. \tag{1}$$

The constant $\lambda \in \mathbb{K}$ is fixed once and for all and is called the weight. This set-up can easily be generalized as, for instance, in [2]. We call the relation (1) the Rota–Baxter relation (RBR) and the operator R is called Rota–Baxter operator (RBO) of weight $\lambda \in \mathbb{K}$. Let us call the tuple $\mathcal{A}_R := (\mathcal{A}; R)$ a Rota–Baxter $\mathbb{K}$-algebra of weight $\lambda \in \mathbb{K}$.

The case $\lambda = 0$:

$$R(x)R(y) = R(R(x)y + xR(y)), \quad x, y \in \mathcal{A} \tag{2}$$

may easily be identified as a shuffle relation as one can see it, for instance, by defining the RBO R as the integral operator on a well chosen function algebra, it then reflects the rule of integration by parts:

$$R[f](x) := \int_0^x f(y)\,\mathrm{d}y. \tag{3}$$

For the case of an arbitrary $\lambda \in \mathbb{K}$ the relation (1) should therefore be regarded as a possible generalization of the shuffle relation. Of great interest is the class of idempotent RBOs ($R = R^2$) on which we will comment in the last section.

For the rest of this Letter we concentrate on the natural case of $\lambda = +1$ which can always be achieved by a normalization of $R \to \lambda^{-1}R,\ \lambda \neq 0$. Nevertheless, we will use the phrase *Rota–Baxter operator of arbitrary weight* λ, but ignore the λs in most of the equations.

As Atkinson showed in [4], a linear operator R from $\mathcal{A}$ to $\mathcal{A}$ satisfying the RBR is equivalent to the fact that $\mathcal{A}$ is a subdirect difference in the sense of Birkhoff of two subalgebras $\mathcal{P}, \mathcal{F} \subset \mathcal{A}$.

Suppose R fulfils the RBR of weight $\lambda = 1$:

$$R(x)R(y) + R(xy) = R(R(x)y + xR(y)), \tag{4}$$

the same is then also true for the 'opposite' operator $R^- := 1 - R$.

We would like to present two examples of Rota–Baxter algebras. The first one was introduced by Miller in [5]. Let $\mathcal{A}$ be a finite-dimensional $\mathbb{K}$-vector space with basis $e_1, \dots, e_n$ and make it into an associative algebra by defining the product componentwise on the column matrices of $n = s + t$ components:

$$a, b \in \mathcal{A}, \quad a_i, b_j \in \mathbb{K}, \qquad a \cdot b = \sum_{i=1}^{n} a_i e_i \cdot \sum_{j=1}^{n} b_j e_j := \sum_{i=1}^{n} (a_i b_i) e_i, \tag{5}$$

Then the following matrix R defines a RBO of weight 1:

$$R := \begin{pmatrix} S_s & 0 \\ 0 & T_t \end{pmatrix}, \tag{6}$$

$$S_s := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}_{s \times s}, \qquad T_t := \begin{pmatrix} 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -1 & \dots & -1 & 0 \end{pmatrix}_{t \times t}. \tag{7}$$

Another example is provided by the algebra of Laurent polynomials. The (idempotent) RBO is now given by the projector R_{ms}:

$$R_{ms}\left(\sum_{i=-m}^{\infty} c_i z^i\right) := \sum_{i=-m}^{-1} c_i z^i, \tag{8}$$

$$x := \sum_{i=-m}^{\infty} a_i z^i, \qquad y := \sum_{j=-n}^{\infty} b_j z^j,$$

$$\begin{aligned} & R_{ms}(R_{ms}(x)y + xR_{ms}(y) - xy) \\ & \quad = R_{ms}\left(\sum_{i=-m}^{-1} a_i z^i \sum_{j=-n}^{\infty} b_j z^j + \sum_{i=-m}^{\infty} a_i z^i \sum_{j=-n}^{-1} b_j z^j - \sum_{i=-m}^{\infty} a_i z^i \sum_{j=-n}^{\infty} b_j z^j\right) \qquad (9) \\ & \quad = R_{ms}\left(\sum_{i=-m}^{-1} a_i z^i \sum_{j=-n}^{-1} b_j z^j\right) - R_{ms}\left(\sum_{i=0}^{\infty} a_i z^i \sum_{j=0}^{\infty} b_j z^j\right) \\ & \quad \overset{(8)}{=} R_{ms}\left(\sum_{i=-m}^{-1} a_i z^i \sum_{j=-n}^{-1} b_j z^j\right) \\ & \quad = R_{ms}(x) R_{ms}(y). \qquad (10) \end{aligned}$$

□

The opposite operator

$$R_{ms}^{-} := 1 - R_{ms},\ \ R_{ms}^{-}\left(\sum_{i=-m}^{\infty} c_i z^i\right) := \sum_{i=0}^{\infty} c_i z^i,$$

is also a RBO. The projector R_{ms} is used in some renormalization procedure of QFT which is called the minimal subtraction scheme and where R_{ms} is a so called renormalization map. It is intimately related to the Riemann–Hilbert problem as was shown in [9, 10].

Replacing the -1 on the right-hand side in (8) by 0 also gives a RBO. For a general $r \in \mathbb{Z}\backslash\{-1, 0\}$, noted by

$$R_r\left(\sum_{i=-m}^{\infty} c_i z^i\right) := \sum_{i=-m}^{r} c_i z^i$$

the RBR (4) does not hold, as one can see by the following argument. For $r > 0$ one can show that one gets on the right-hand side of (10) a polynomial of order $2r$ whereas on the left-hand side one only gets a polynomial of order r. For $r < -1$ the same argument applies.

A more detailed presentation of Rota–Baxter algebras will be given elsewhere.

Inspired by the work of Semenov-Tian-Shansky [15] we define now the following new operator on the Rota–Baxter algebra $\mathcal{A}_R$, R of weight λ:

$$B_\lambda := \lambda - 2R, \tag{11}$$

which we will call modified Rota–Baxter operator of weight λ and which fulfils the relation:

$$B_\lambda(x)B_\lambda(y) = B_\lambda(B_\lambda(x)y + xB_\lambda(y)) - \lambda^2 xy, \tag{12}$$

$$\begin{aligned} B_\lambda(x)B_\lambda(y) &= \lambda^2 xy - 2\lambda(R(x)y + xR(y)) + 4R(x)R(y) \\ &\overset{(4)}{=} (\lambda - 2R)(2\lambda xy - 2R(x)y - 2xR(y)) - \lambda^2 xy \\ &= B_\lambda(B_\lambda(x)y + xB_\lambda(y)) - \lambda^2 xy. \end{aligned}$$

□

Equation (12) is called the modified Rota–Baxter relation. As we already said before, when normalizing the RBO of weight $\lambda \neq 0$ to $\lambda^{-1}R$ it holds the RBR (4). The operator $B := 1 - 2R$ then fulfils the relation

$$B(x)B(y) = B(B(x)y + xB(y)) - xy. \tag{13}$$

Whereas the opposite operator $\tilde{B} := 1 + 2R$ fulfils (13) if the RBO R is of weight $\lambda = -1$.

In the Lie algebraic context of [15] this relation is called (the operator form of the) modified classical Yang–Baxter equation. Let us remark here that in a Lie algebraic context the relation (4) is called (operator form of the) classical Yang–Baxter equation [16]. Since we work here in the realm of associative algebras we will call expression (13) the modified associative classical Yang–Baxter relation (maCYBR).

We follow hereby the terminology introduced by Aguiar in [13, 14] who defined on an associative algebra $\mathcal{A}$ an associative analog of the classical Yang–Baxter equation for the r-matrix $r \in \mathcal{A} \otimes \mathcal{A}$:

$$aCYBE(r) := r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0. \tag{14}$$

We would like to underline here that the RBR (4) with respect to the equation (14) may be interpreted in the same way as it is done in the Lie algebraic context of the CYBE in [16]. This interesting link of the (associative and modified associative) classical Yang–Baxter relation to the realm of Rota–Baxter operators is part of work in progress [17].

3. The Dendriform Di- and Trialgebra

We will give here the definitions of a dendriform di- and trialgebra following the work of Loday and Loday and Ronco [11, 12]. Let $\mathcal{A}$ be a $\mathbb{K}$-vector space equipped with the following two binary compositions:

$$\substack{\prec \\ \succ}: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$

which are supposed to hold the so called dendriform dialgebra relations [11]:

$$(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c), \tag{15}$$

$$a \succ (b \prec c) = (a \succ b) \prec c, \tag{16}$$

$$a \succ (b \succ c) = (a \prec b) \succ c + (a \succ b) \succ c. \tag{17}$$

The triple $(\mathcal{A}, \prec, \succ)$ is then called a dendriform dialgebra.

We now come to the trialgebra structure. As before, let $\mathcal{A}$ be a $\mathbb{K}$-vector space equipped with the three binary compositions:

$$\substack{\prec \\ \succ}: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \qquad \cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$

which are supposed to satisfy the following relations, the so-called dendriform trialgebra axioms [12]:

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c + b \cdot c), \tag{18}$$

$$(a \succ b) \prec c = a \succ (b \prec c), \tag{19}$$

$$a \succ (b \succ c) = (a \prec b + a \succ b + a \cdot b) \succ c, \tag{20}$$

$$(a \prec b) \cdot c = a \cdot (b \succ c), \tag{21}$$

$$(a \succ b) \cdot c = a \succ (b \cdot c), \tag{22}$$

$$(a \cdot b) \prec c = a \cdot (b \prec c), \tag{23}$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c). \tag{24}$$

We also define a fourth multiplication on $\mathcal{A}$:

$$*: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \quad a * b := a \prec b + a \succ b + a \cdot b \tag{25}$$

which shows to be an associative binary composition on $\mathcal{A}$:

$$
\begin{aligned}
(a * b) * c &= (a \prec b + a \succ b + a \cdot b) * c \\
&\overset{(18,20,21)}{=} a \prec (b * c) + a \succ (b \succ c) + a \cdot (b \succ c) + \\
&\quad + (a \succ b) \prec c + (a \succ b) \cdot c + (a \cdot b) \prec c + (a \cdot b) \cdot c \\
&\overset{(19,22,23)}{=} a \prec (b * c) + a \succ (b * c) + a \cdot (b \succ c + b \prec c) + (a \cdot b) \cdot c \\
&\overset{(24)}{=} a * (b * c).
\end{aligned} \tag{26}
$$

We will not go into any details with respect to these algebraic structures which can be found in the before-mentioned literature. Loday *et al.* give several examples of dendriform di- and trialgebras in [11, 12].

In the following section we will show that Rota–Baxter operators of arbitrary weight λ respectively Rota–Baxter algebras provide another class of interesting examples for these two algebraic structures.

4. Dendriform Di- and Trialgebra Structures on Rota–Baxter Algebras

In [13], Aguiar observed that on a Rota–Baxter algebra with RBO of weight $\lambda = 0$, $(\mathcal{A}; R)$, the following two binary compositions $\substack{\prec \\ \succ} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $a \prec b := aR(b)$ and $a \succ b := R(a)b$ fulfil the dendriform dialgebra relations (15–17) and therefore $(\mathcal{A}, \prec, \succ)$ is a $\mathbb{K}$-dendriform dialgebra.

This observation may be extended to general Rota–Baxter algebras with a RBO of arbitrary weight $\lambda \in \mathbb{K}$ by using the modified Rota–Baxter operator $B_\lambda = \lambda - 2R$. Then the two binary compositions:

$$
\substack{\prec \\ \succ} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \quad \begin{array}{l} a \prec b := aB(b) - \lambda ab \\ a \succ b := B(a)b + \lambda ab \end{array} \tag{27}
$$

make $(\mathcal{A}, \prec, \succ)$ a dendriform dialgebra as we will show now. We mentioned above that one can always normalize the RBO R to $\hat{R} := \lambda^{-1}R$, $\lambda \neq 0$ to fulfil the RBR for the weight $+1$ so that the operator $B = 1 - 2R$ holds the relation (13). Of course we could have defined them also by the two RBOs R and R^{-}, $a \prec b = -2aR(b)$ and $a \succ b := 2R^{-}(a)b$, $R^{-} := 1 - R$★.

$$
\begin{aligned}
(a \prec b) \prec c &= aB(b)B(c) - abB(c) - aB(b)c + abc \\
&\overset{(13)}{=} aB(B(b)c) + aB(bB(c)) - abB(c) - aB(b)c + abc - abc + \\
&\quad + aB(bc) - aB(bc) \\
&= a \prec (bB(c) - bc) + a \prec (B(b)c + bc) \\
&= a \prec (b \prec c) + a \prec (b \succ c), \\
a \succ (b \succ c) &= B(a)B(b)c + B(a)bc + aB(b)c + abc \\
&\overset{(13)}{=} (B(B(a)b) + B(aB(b) - ab)c + B(a)bc + aB(b)c + abc \\
&= (a \prec b) \succ c + (a \succ b) \succ c.
\end{aligned}
$$

□

★From this point of view it would be more convenient to use a RBO of weight $\lambda = 2$.

Relation (16) follows by associativity of $\mathcal{A}$. We mention as an aside that for an idempotent RBO the RBR (4) is fulfilled on both compositions $(\prec, \succ)$:

$$R(x) \substack{\prec \\ \succ} R(y) + R(x \substack{\prec \\ \succ} y) = R(R(x) \substack{\prec \\ \succ} y + x \substack{\prec \\ \succ} R(y)) \tag{28}$$

Regarding the result of Aguiar it seems to us that the most natural dendriform structure on a Rota–Baxter algebra of weight $\lambda \neq 0$ is the trialgebra one. On the Rota–Baxter algebra $(\mathcal{A}; R)$ we denote the multiplication now by $a \cdot b \in \mathcal{A}$, $a, b \in \mathcal{A}$. The RBO is supposed to be of weight $\lambda = -1$ just for reasons of clarity with regard to the trialgebra axioms in (18–24).

We define the following two binary compositions:

$$\substack{\prec \\ \succ} : \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \quad \begin{array}{l} a \prec b := a \cdot R(b), \\ a \succ b := R(a) \cdot b. \end{array} \tag{29}$$

The fourth composition looks as follows:

$$* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \ \ a * b := a \cdot R(b) + R(a) \cdot b + a \cdot b. \tag{30}$$

We show now that these multiplications hold the axioms in (18)–(24) and that (30) is associative. This makes $(\mathcal{A}, \prec, \succ, \cdot)$ a dendriform trialgebra.

Relations (19) and (21)–(24) are easy to show by the associativity of the Rota–Baxter algebra. The axioms (18,20) follow from the RBR (4):

$$\begin{aligned} (a \prec b) \prec c &= a \cdot R(b) \cdot R(c) \\ &\overset{(4)}{=} a \cdot R(R(b) \cdot c + b \cdot R(c) + b \cdot c) \\ &= a \prec (b \succ c + b \prec c + b \cdot c), \\ a \succ (b \succ c) &= R(a) \cdot R(b) \cdot c \\ &\overset{(4)}{=} R(R(a) \cdot b + a \cdot R(b) + a \cdot b) \cdot c \\ &= (a \succ b + a \prec b + a \cdot b) \succ c. \end{aligned}$$

We are left to show the associativity of the fourth composition $*\colon \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$:

$$\begin{aligned} (a * b) * c &= (R(a) \cdot b + a \cdot R(b) + a \cdot b) * c \\ &\overset{(4)}{=} R(a) \cdot (R(b) \cdot c + b \cdot R(c) + b \cdot c) + \\ &\quad + a \cdot R(R(b) \cdot c + b \cdot R(c) + b \cdot c) + \\ &\quad + a \cdot (R(b) \cdot c + b \cdot R(c) + b \cdot c) \\ &= a * (R(b) \cdot c + b \cdot R(c) + b \cdot c) \\ &= a * (c * b) \end{aligned}$$

□

Rota–Baxter algebras may thus be equipped with dendriform di- and trialgebra structures. It is interesting to see this dendriform structures on Rota–Baxter algebras with respect to Atkinson's theorem mentioned above.

5. Summary and Outlook

We presented briefly the concept of Rota–Baxter algebras and introduced some new operators which we called modified Rota–Baxter operator. It was shown that these operators fulfil a new equation which we called the modified associative classical Yang–Baxter relation. We mentioned also the intimate relation of the Rota–Baxter relation and the associative analog of the classical Yang–Baxter equation.

After giving the definitions of dendriform di- and trialgebras we used the modified Rota–Baxter operators to extend the result of Aguiar to associative $\mathbb{K}$-algebras equipped with a Rota–Baxter operator of arbitrary weight λ. The trialgebra structure shows to be the most natural dendriform structure on a Rota–Baxter algebra of arbitrary weight.

The structure implied on an associative $\mathbb{K}$-algebra by the general Rota–Baxter relation is quite astonishing and requires further investigations. We remark here briefly that on a Rota–Baxter algebra equipped with the associated Lie-bracket the modified classical Yang–Baxter relation is fulfilled and one can define a pre-Lie structure related to the Jordan product and the modified Rota–Baxter operator.

The Rota–Baxter relation is also of importance with respect to multiple-zeta-values (MZVs) and a possible q-deformation of them. All these structures relate to quantum field theory [9, 10, 18] and we expect a further study of Rota–Baxter operators to be of significance for the understanding of such theories.

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