# Exercise-sheet 2 (23rd-24th of april)

## 1 In class exercise:

### 1.1 Properties of hermitian and unitary operators

For a linear operator $\widehat{A}$ the hermitian operator $\widehat{A}^\dagger$ is defined by: $\langle u|\widehat{A}^\dagger|v\rangle = \langle v|\widehat{A}|u\rangle^*$. An operator is hermitian if $\widehat{A} = \widehat{A}^\dagger$, anit-hermitian if $\widehat{A} = -\widehat{A}^\dagger$, and unitary if $\widehat{A}^\dagger = \widehat{A}^{-1}$. A commutator is defined as $[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$. Show:

- $(\widehat{A} + \widehat{B})^\dagger = \widehat{A}^\dagger + \widehat{B}^\dagger = \widehat{B}^\dagger + \widehat{A}^\dagger$
- $(\widehat{A}^\dagger)^\dagger = \widehat{A}$
- $(\lambda\widehat{A})^\dagger = \lambda^*\widehat{A}^\dagger$
- $(\widehat{A}\widehat{B}...\widehat{Z})^\dagger = \widehat{Z}^\dagger...\widehat{B}^\dagger\widehat{A}^\dagger$
- $[\widehat{A}, \widehat{B}]^\dagger$is anti-hermitian, if $\widehat{A}$ and $\widehat{B}$ are hermitian.
- for the matrix elements of the hermitian operator the following condition applies: $a_{ij} = a_{ji}^*$
- if $\widehat{A}$ is hermitian, then $e^{i\widehat{A}}$ is unitary
- the eigenvalues of a hermitian operator are real and the eigenvectors to different eigenvalus are orthogonal.
- the eigenvalues of a unitary operator are on the unit-circle.
- if $\widehat{A}$ and $\widehat{B}$ are hermitian, and the commutator vanish, and $\overrightarrow{a}$ is eigenvector of $\widehat{A}$. Show that then $\widehat{B}\overrightarrow{a}$ is eigenvector of $\widehat{A}$.

### 1.2 State in real space

A state $|\psi\rangle$ of a particle is described in real space by the wavefunction:

$\langle x|\psi\rangle = \alpha e^{-\beta|x-x_0|+i\gamma x}$,$\beta > 0$,$\gamma > 0$, $x_0 > 0$

Normalize $\langle x|\psi\rangle$ and draw $|\langle x|\psi\rangle|^2$. Calculate $\langle x\rangle$ the probabilty that the particle is in the interval a) $(-\infty, 0]$, b) $[0, x_0]$ and c) $(x_0, \infty)$. Transform $|\psi\rangle$ to momentum space by calculating:

$\langle p|\psi\rangle = \int_{-\infty}^{\infty} dx\langle p|x\rangle\langle x|\psi\rangle$ with $\langle p|x\rangle = e^{-i\frac{p}{\hbar}x}$

Calculate $\Delta x\Delta p$ for the particle in state $|\psi\rangle$.

# 2 Homework - due date: 6th of May 2009 (40 points)

## 2.1 Delta-distribution (8 points)

Show that for $n \to \infty$ the two series

$f_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$ and $g_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}$

have the properties, defining a $\delta$-distribution.

## 2.2 Dot-product (12 points)

For $\phi_1$, $\phi_2 \in S(\mathbb{R}^{\mathrm{n}})$(Schwartz-space) a dot-product is defined as:

$\langle\phi_1|\phi_2\rangle = \int d^nx\phi_1^*(\vec{x})\phi_2(\vec{x})$.

- show that $\langle.|.\rangle$ is a hermitian dot-product.
- show that the operator $i\vec{\nabla}$ is hermitian with respect to $\langle.|.\rangle$.
- show that the fouriertransform with respect to $\langle.|.\rangle$ is unitary. Therefore $\langle\mathcal{F}\phi_1|\phi_2\rangle = \langle\phi_1|\mathcal{F}^{-1}\phi_2\rangle$ and $\langle\phi_1|\phi_2\rangle = \langle\mathcal{F}\phi_1|\mathcal{F}\phi_2\rangle$ is valid.
- show that the expectation-value $\langle A\rangle = \langle\psi|\widehat{A}|\psi\rangle$ of hermitian operators are real.
- show that the statistical dispersion $(\triangle\ A)^2 = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle$ of a hermitian operator $\widehat{A}$ vanishes if $\psi$ is an eigenfunction of $\widehat{A}$ .
- A operator $\widehat{A}$ which is equal to his hermitian operator $\widehat{A}^\dagger$, but for which the domain of the two operators is different is called self-adjoint. Show that $\widehat{p}_r = \frac{1}{ir}\frac{\partial}{\partial r}r$ in $\mathbb{R}^3$ is self-adjoint.

## 2.3 Calculus with operators (10 points)

Given is the Hamiltonoperator and another operator:

$$\widehat{H} = -\hbar\omega\frac{1}{2}\begin{pmatrix} 1 & -i\sqrt{3} \\ i\sqrt{3} & -1 \end{pmatrix}, \widehat{\sigma}^{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- Calculate $[\widehat{H}, \widehat{\sigma}^x]$.
- Give the eigenvectors and eigenvalues of the operators.
- Give the time-evolution of the eigenstates of the Hamiltonoperator. At $t = 0$ the system is in a certain state $\psi(t = 0)$, which is eigenstate to the positive eigenvalue of $\sigma^x$. What is the state $\psi(t)$ and the expectationvalue of $\langle\sigma^x\rangle$ for $t > 0$.

## 2.4 "Law of probability conservation" (10 points)

If the normalization relation $\int d^3x\psi^*\psi = 1$ is interpreted in the sense of probability theory, to find a particle in the element $d^3x$. Then there must be a conservation law. Derive this and give a classical interpretation. (Hint: equation of continuity).