Exercise-sheet 3 (7th.-8th. of May)

1 In class exercise:

1.1 The harmonic oscillator

As you know already from the lecture, the hamiltonoperator for the 1-dimensional harmonic oscillator with mass m and frequency ω is:

$$\widehat{H} \; = \; -\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2 x^2$$

- Show that the stationary Schrödinger-equation $\widehat{H}\psi(x) = E\psi(x)$ with $b = \sqrt{\frac{\hbar}{m\omega}}$, $y = \frac{x}{b}$, $\epsilon = \frac{2E}{\hbar\omega}$ and $\psi(x) = u(\frac{x}{b})$ takes on the following dimensionless form:
  $\frac{d^2u}{dy^2} + (\epsilon - y^2)u = 0$
  b is there the charataristic length-scale.

- Assuming $\phi(\vec{r})$ is an eigenfunction of a Hamiltonoperator with eigenvalue E. Show that $\phi(\vec{r}, t) = e^{-\frac{i}{\hbar}Et}\phi(\vec{r})$ solves the Schröding-equation. Which condition must be valid for $\phi(\vec{r})$ so that a quantummechanical interpretation is possible?

- The ladderoperators $\widehat{a}^\dagger$ and $\widehat{a}$ are defined as following:
  $\widehat{a} = \frac{1}{\sqrt{2}}(\frac{\widehat{x}}{b} + \frac{ib}{\hbar}\widehat{p})$, $\widehat{a}^\dagger = \frac{1}{\sqrt{2}}(\frac{\widehat{x}}{b} - \frac{ib}{\hbar}\widehat{p})$
  Show that:
  $\widehat{H} = \hbar\omega(\widehat{a}^\dagger\widehat{a} + \frac{1}{2})$

- Calculate:
  $[\widehat{a}, \widehat{a}^\dagger]$, $[\widehat{H}, \widehat{a}^\dagger]$, $[\widehat{H}, \widehat{a}]$, $[\widehat{n}, \widehat{a}]$ and $[\widehat{n}, \widehat{a}^\dagger]$ - with $\widehat{n} = \widehat{a}^\dagger\widehat{a}$ the number operator.

- Calculate the normalized solution $\phi_0(x)$ of the equation $\widehat{a}\phi_0 = 0$. What is $\widehat{H}\phi_0$?

- Define $\phi_n = \frac{1}{\sqrt{n!}}(\widehat{a}^\dagger)^n\phi_0$ and calculate $\widehat{H}\phi_n$, $\int |\phi_n(x)|^2 dx$ and $\widehat{a}\phi_n$. $\widehat{a}^\dagger$ is called rasingoperator, $\widehat{a}$ is called loweringoperator. Why?

- Show that the energy-eigenvalues are positive and conclude further that besides the constructed ϕ_n with the eigenvalues $E_n = \hbar\omega(n + \frac{1}{2})$ there are no other eigenfunctions.

2 Homework - due date: 13th of May 2009 (32 points)

2.1 Revision (8 points)

- Check which of the following operators is hermitian: $\vec{\nabla}$, $\frac{\hbar}{i}\vec{\nabla}$, and$\vec{\nabla}^2$ (each one point).
- Show: $[A+B,C]=[A,C]+[B,C]$, $[AB,C]=A[B,C]+[A,C]B$ and $[A[B,C]]+[B[C,A]]+[C,[A,B]]=0$ (each 1 point).
- Show: if $[A,B]=c$ with $c\epsilon\mathbb{C}$ then $[A^n,B]=cnA^{n-1}$ (2 points).

2.2 Harmonic oscillator: Continuation of the in-class exercise (14 points)

- The Hermite-polynomials $H_k(y)$ are the defined through the following expansion:
  $e^{-z^2+2zy}=\sum_{k=0}^{\infty}\frac{z^k}{k!}H_k(y)$ for $z\epsilon\mathbb{C}$, $y\epsilon\mathbb{R}$.
  Show:
  $H_k(y)=(-1)^k e^{y^2}\frac{d^k}{dy^k}e^{-y^2}$ and $H_k'(y)=2yH_k(y)-H_{k+1}(y)$.
- Show:
  $\phi_n(x)=\frac{1}{\sqrt{b2^n n!\sqrt{\pi}}}e^{-\frac{1}{2b^2}x^2}H_n(\frac{x}{b})$.
- Show that the Hermite-polynomials are complete by showing $\langle H_k|f\rangle=0\forall k\Rightarrow f=0$. Consider the function: $F(z)=\int e^{-z^2+2zx}f^*(x)dx$ for $z\epsilon\mathbb{C}$.
- Sketch the ground-state (ϕ_0), and the first two excited states (ϕ_1 and ϕ_2).
- Show that $\langle p\rangle$ and $\langle x\rangle$ vanish in their eigenstates.
- Calculate Δx and Δp in the eigenstate ϕ_n and show $\Delta x\Delta p=(n+\frac{1}{2})\hbar$.
- What is the lowest possible energie for $\langle p\rangle=\langle x\rangle=0$?

2.3 Plane waves (10 points)

Find the solution of the one-dimensional time-dependend Schrödinger-equation for the case $V(x,t)=0$. Discuss the physical significance of the solutions.