Exercise-sheet 7 (10th and 12th of June)

1 In class exercise:

1.1 Exchange-Hamiltonian

Between two spin-1/2 particles there is an interaction described by the following hamltonian: $H = JS_1S_2$, with $S_i = \frac{\hbar}{2}\sigma$.

1.1. Obtain the matrix-elements of H.

1.2. Obtain the eigenvalues and eigenfunctions.

1.3. Calculate the expectation-value of $(S_1 + S_2)^2$ for the eigenvalues.

1.2 Three particles with spin 1/2

Obtain the eigenvectors in the 2^3-dimensional Hilbert-space of the three particles with spin 1/2.

1.3 N-Spins

Consider now a system of N spin-1/2 particles (e.g. electrons in a crystal). Which values are possible for the total spin ($\sum_{i=1}^{N} \vec{s_i}$) and how many different mulipletts exists for a given total-spin?

2 Homework - due date 17th of June 2009 (32 points)

2.1 Connection-conditions (8 points)

Derive the connection-conditions for the derivation of the eigenstate $\partial_x \Psi(x)$ for the following discontinous potentials:

2.1. $V(x) = V_0 \Theta(x)$ (what happens for $V_0 = \infty$?)

2.2. $V(x) = \frac{\hbar^2 \lambda}{2m} \delta(x)$

Hint: Write $\Psi'(+\epsilon) - \Psi'(-\epsilon)$ in an intergral over the discontinuity and evalidate it with the help of the Schrödingerequation.

2.2 Addition of angular momentum (14 points)

If one considers the angular momentum of two particles, then there are two complete sets of commuting angular-momentum operators: $(J^{(1)})^2$, $J_3^{(1)}$, $(J^{(2)})^2$, $J_3^{(2)}$ with the eigenfunctionbasis $|j_1 m_1 j_2 m_2\rangle \equiv |j_1 m_2\rangle \otimes |j_2 m_2\rangle$ and $(J^{(1)})^2$, $(J^{(2)})^2$, J^2, J_3 with the eigenfunctionbasis $|j_1 j_2 j m\rangle$, with $J = J^{(1)} \otimes 1 + 1 \otimes J^{(2)}$ the total angular momentum. The corresponding coefficients (Clebsch/Gordan-coefficients) of the two bases are scalar-products of the form $\langle j_1 m_1 j_2 m_2 | j_1 j_2 j m\rangle$.

2.1. Show that the scalar producets do not vanish only if $m = m_1 + m_2$.

2.2. Why is $j_1 + j_2$ the maximum value of j? Show through counting of the states that $j \geq |j_1 - j_2|$ holds. Which values are possible for j?

For maximum value of m and m_1 one can require the coefficient to be positive: $\langle j_1 j_1 j_2 (j - j_1) | j_1 j_2 j j\rangle \geq 0$.

2.3. Show that the states with maximum value of j and m are given by $|j_1 j_2 j = j_1 + j_2 m = \pm j\rangle = |j_1 \pm j_1\rangle \otimes |j_2 \pm j_2\rangle$ meaning the coefficient is equal to one.

Consider the addition of two angular momenta with $j_1 = j_2 = 1$.

2.4. Starting from the state obtained in 3. with $j = m = 2$ obtain the other states with $j = 2$ by appling the ladderoperators.

2.5. Find the state with $j = m = 1$, by constructing a state with $m = 1$ which is orthogonal to the state with $j = 2, m = 1$. Verify that this state really has a total momentum $j = 1$.

2.6. Starting from the state obtained in 5. with $j = m = 1$, calculate the other states with $j = 1$ by applying the ladderoperators.

2.7. Find the state with $j = m = 0$, by constructing a state with $m = 0$ which is orthogonal to the states with $j = 2, m = 0$ and $j = 1, m = 0$. Verify that this state has indeed $j = 0$.

2.3 Spin-1/2 particle in a magnetic field (10 points)

Consider a particle with spin-1/2 in an external magnetic field $\overrightarrow{B}$. The spin-dependent part of the Hamiltonian is:

$$H_{mag} = -g\mu\vec{B}\frac{\hbar}{2}\vec{\sigma}$$

At time t=0 the particle is in a spin-state $\chi_S(t = 0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Calculate $\chi(t)$ and the expectation value of the spin $\vec{S}$ for any arbituary time t.