Mock-exam

Purpose of this mock-exam is it, to give you an impression what the actual exam might look like. Try to solve the exam at home, without using books/notes, and check how much you manage to finish within 3 hours. In our last tutorial you may discuss with your tutor examples you found difficult. At the exam you have to reach 50 % of the points to pass, for the mock-exam here, that would correspond to 90 points.

1 Questions about QM (10+2+6+4+4+7+4+4=41 points)

1.1. Write down the probarbility-density and currenct-density of a wave-function. Show that the equation of continuity is valid, and conclude the conservation of probarbility from it.

1.2. Is it possible, that two electrons are in a state with the follwoing position-wave-function$\psi(\vec{r}^{(1)}, \vec{r}^{(2)}) = A\exp(-\frac{1}{b^2}(|\vec{r}^{(1)}|^2 + |\vec{r}^{(2)}|^2))$? Explain your answer.

1.3. What are the energie-eigenvalues of the hydrogen-atom? Whats the dengeneracy? Which form have the eigenfunctions? Sketch the radial-wavefunctions for the states of the first and second main-shell.

1.4. A wave-function be real. Show that then the expectation-value of the momentum vanishes.

1.5. Show: If an operator commutes with two components of the angular-momentum $\vec{L}$, then it commutes as well with the third one. Is that true for the spin too?

1.6. A particle of mass m and charge q moves in a magnetic field $\vec{B}$ with vectorpotential $\vec{A}$ resp. $\vec{A}'$. Show how the wave-functions of the Schrödingerequation with $\vec{A}$ have to be transformed, so that they become those with $\vec{A}'$.

1.7. Show: If $[A, B] = c$ with $c\epsilon\mathbb{C}$ then $[A^n, B] = cnA^{n-1}$.

1.8. Show: If $\widehat{A}$ is hermitian, then $e^{i\widehat{A}}$ is unitary.

2 Harmonic Oscillator (6+8+8=22 points)

2.1. What are the energy-eigenvalues of the one-dimensional harmonic oscillator? Which form and parity do the eigenfunctions have? Sketch the wavefunction of the three lowest states.

2.2. Consider the expectation value of the kinetic energy T and of the potential energy V in the energy-eigenstates $|n\rangle$ of the one-dimensional harmonic oscillator and show $\langle n|T|n\rangle = \langle n|V|n\rangle$.

2.3. A particle is in the ground-state of the one-dimensional harmonic oscillator. Calculate the probarbility that the particle is outside of the classically allowed area. Discuss the result.

3 Delta-potential (6+10=16 points)

We are looking for the bound states of the one-dimensional potential: $V(x) = -\frac{\lambda\hbar^2}{2m}\delta(x)$.

3.1. What are the connection-conditions of the wave-function?

3.2. With the help of (1) derive the equation for the energy, and solve it.

4 Squeezed States (5+8+5=18 points)

4.1. Show $(e^{\alpha x \frac{\partial}{\partial x}} f)(x) = f(e^{\alpha} x)$. Derive at both sides with respect to α for the proof.

4.2. Consider for the one-dimensional harmonic oscillator the normalized state $|\psi_\alpha\rangle = C_\alpha e^{\frac{\alpha}{2}((a^\dagger)^2 - a^2 - 1)}|\phi\rangle$, where $\langle\phi|\phi\rangle = 1$. Use (1) and a and $a^\dagger$ in real-space to express $\psi_\alpha(x)$ through $\phi(x')$, in other words find $x'(x)$. Why is $|\psi_\alpha\rangle$ called "squeezed"?

4.3. Show that $S := e^{z(a^\dagger)^2 - z^* a^2}$ for $z\epsilon\mathbb{C}$ is unitary. What hold for C_α then?
Hint $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$

5 Kramers degeneracy (3+8+4+7=22 points)

Consider the hamiltonian for a spin $\frac{1}{2}$ particle: $H = \frac{1}{2m}\vec{p}^{\,2} + V(\vec{r}) + W(\vec{r})\vec{L}\vec{S}$.

5.1. Show that for the Pauli-matrices $\sigma_k^* = -\sigma_2\sigma_k\sigma_2$ holds, with k as an index.

5.2. Show that: $\psi'(t) = \sigma_2\psi^*(-t)$ solves the Schrödingerequation, if $\psi(t)$ is a solution.

5.3. Show that for the mapping $\psi \mapsto \psi'$ $\langle\psi_1'|\psi_2'\rangle = \langle\psi_2|\psi_1\rangle$ is valid.

5.4. Assuming ψ_E be a solution of the stationary Schrödingerequation to the energy E. Show that $\psi_E' = \sigma_2\psi_E^*$ is a solution to the same energy and $\langle\psi_E|\psi_E'\rangle = 0$ is valid.

6 Quantummechanical rotor in electric field (2+6+7+9=24 points)

The hamiltonian of a static rotor with moment of inertia I is $H_0 = \frac{1}{2I}\vec{L}^2$. Hint: $Y_{10}(\vartheta,\phi) = \sqrt{\frac{3}{4\pi}} cos\vartheta$, $Y_{20}(\vartheta,\phi) = \sqrt{\frac{5}{4\pi}}(\frac{3}{2}cos^2\vartheta - \frac{1}{2})$.

6.1. What are the eigenvalues and eigenfunctions of H_0 ?

6.2. The rotor be charged and is exposed to an electric field. This leads to an additional potential of $H' = \frac{az}{r}$. Show, which matrix-elements of H' do not vanish. Which order in the perturbation-theory corrects the energies?

6.3. The rotor is now exposed to another electric field, this leads to an additional potential of $H' = b\frac{2z^2 - x^2 - y^2}{r^2}$. To simplify matters, the perturbation-theory shall be limited to the two lowest energy-levels. Show which matrix-elements (of H') between those eigenstates for those energy-levels, do not vanish.

6.4. Obtain from (3) the relative energy-splittung because of H', which one expects from perturbation-theory of first order. How does the wave-function change?

7 Clebsch/Gordan-coefficients (16 points)

Perform the agular-momentum addition explicitly for $j_1 = j_2 = \frac{1}{2}$. Do this by means of the ladder-operators starting from the state $|\frac{1}{2}, m_1\rangle \otimes |\frac{1}{2}, m_2\rangle$.

8 Two-level system (5+10+4+2=21 points)

Consider a particle with spin $\frac{1}{2}$ in a homogenious magnetic field $\vec{B} = B\vec{e_z}$.

8.1. Which hamiltonian describes the time-evolution of that spin in the magnetic field? Only the interaction with the magnetic field shall be considered here, the kinetic energy of the particle is ignored. Calculate the time-eveolution operator explicitly.

8.2. Take as quantization-axis the z-direction. Express the eigen-states of the spinoperators S_x and S_y through the eigen-states of the operator S_z .

8.3. At time $t = 0$ the system is in eigen-state $|\chi\rangle$ of S_x. Give an expression for $|\chi(t)\rangle$ with the help of (1) and (2). Discuss the solution.

8.4. What changes if we consider the angular-momentum instead of the spin?